Differential Equations and Geosystems

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Abstract

In this paper, we systematize the conjectures of the TJHSST geosystems textbook. We solve an elementary linear first-order series of ordinary differential equations and interpret the solution and its relevance to the study of Earth systems.

1 Introduction

Why do we combine the study of differential equations with the study of the Earth? Simply put, differential equations provide an elegant model for many natural phenomena. The link between the two is the study of ***systems***, giving "geosystems" its name.

> Because most natural systems are time-dependent, their behavior must be described by differential equations. Differential equations are beyond the level of most readers of this book; however, readers who have the required mathematical background are invited to follow the discussion below.
>
> — L. Kump, J. Kasting, and R. Crane, *The Earth System*, 3rd ed. 2010

In order to facilitate the connection between differential equations and systems, we introduce a pet two-component system and its corresponding model. Although the system is simple, it will elucidate general results about the theory of systems and of stability.

> Suppose we have a system of two reservoirs whose states (e.g., amounts of material in the reservoirs) are represented by the variables $A(t)$ and $B(t)$, which are coupled in a feedback loop. Furthermore, suppose that an equilibrium state exists in this system, in which the reservoir sizes are denoted by A_{eq} and B_{eq}. We are interested in how these reservoirs will respond to a disturbance from their equilibrium state. This system can be described by the two following differential equations:
>
> $$dA/dt = a(B - B_{\text{eq}})$$
> $$dB/dt = b(A - A_{\text{eq}})$$
>
> — [1], page 25

Note that this system corresponds to our intuition about systems. If B is at equilibrium, then there is no change in A. If B is greater than the equilibrium, then A will increase if there is a positive link from B to A, represented by the sign of a. If B is lower, then A will decrease. The same is true for the connection between A and B.

> Here, a and b are constants. The feedback loop is positive if both a and b are positive or if both constants are negative. If a and b have opposite signs, the feedback loop is negative. This follows from our definition of positive and negative couplings. A coupling is positive if component A responds in the same direction as the perturbation to component B; it is negative if the response is in the opposite direction.

2 Analysis

Recall we have the system of differential equations:

$$\frac{dA}{dt} = a(B - B_{\text{eq}})$$
$$\frac{dB}{dt} = b(A - A_{\text{eq}})$$

We first suppose the system starts at some initial time $t_0 = 0$, and that $A_0 = A(t_0)$ and $B_0 = B(t_0)$ representing the initial amounts in each reservoir. We then put the initial value problem into *matrix form*, encapsulating both functions into one vector:

$$\vec{X} = \begin{pmatrix} A \\ B \end{pmatrix}, \; \vec{X}' = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \vec{X} + \begin{pmatrix} -aB_{\text{eq}} \\ -bA_{\text{eq}} \end{pmatrix}, \; \vec{X}(t_0) = \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}$$

To solve this system, we first ignore the forcing term and solve the resulting homogeneous system. To solve the homogeneous system, we find the eigenvalues of the matrix. Looking at the characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & a \\ b & -\lambda \end{vmatrix} = \lambda^2 - ab = 0 \implies \lambda = \pm\sqrt{ab}$$
$$\text{Letting } \alpha = \sqrt{ab},\ \lambda = \pm\alpha$$

We now assume $ab \geq 0$, that is, α is real and the system represents a positive feedback loop. We will see what happens in the negative case later.

For $\lambda_1 = \alpha$, we find its associated eigenvector:

$$\begin{pmatrix} -\alpha & a \\ b & -\alpha \end{pmatrix} \vec{K}_1 = \vec{0} \to -\alpha k_1 + a k_2 = 0 \to k_1 = \frac{a k_2}{\alpha}$$

Because $\frac{a}{\alpha} = \frac{a}{\sqrt{ab}} = \frac{\sqrt{a}}{\sqrt{b}}$, let $\beta = \sqrt{\frac{a}{b}}$ so $k_1 = \beta k_2$. Taking $k_2 = 1$,

$$\vec{K}_1 = \begin{pmatrix} \beta \\ 1 \end{pmatrix} \text{ so } \vec{X}_1(t) = \begin{pmatrix} \beta \\ 1 \end{pmatrix} e^{\alpha t}$$

Similarly, for $\lambda_2 = -\alpha$:

$$\begin{pmatrix} \alpha & a \\ b & \alpha \end{pmatrix} \vec{K}_1 = \vec{0} \to \alpha k_1 + a k_2 = 0 \to k_1 = -\frac{a k_2}{\alpha} = -\beta k_2$$

Taking $k_2 = 1$ again,

$$\vec{K}_2 = \begin{pmatrix} -\beta \\ 1 \end{pmatrix} \text{ so } \vec{X}_2(t) = \begin{pmatrix} -\beta \\ 1 \end{pmatrix} e^{-\alpha t}$$

The complementary solution to the homogenous equation is therefore

$$\vec{X}_c(t) = c_1 \begin{pmatrix} \beta \\ 1 \end{pmatrix} e^{\alpha t} + c_2 \begin{pmatrix} -\beta \\ 1 \end{pmatrix} e^{-\alpha t}$$

We now solve for a particular solution by the method of undetermined coefficients. Because the forcing term is a constant vector, we assume the form $\vec{X}_p = \begin{pmatrix} E \\ F \end{pmatrix}$, where E and F are "undetermined coefficients", or constants whose values will be determined by the differential equation.

$$\vec{X}_p' = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \vec{X}_p + \begin{pmatrix} -aB_{\text{eq}} \\ -bA_{\text{eq}} \end{pmatrix}$$

Because the derivative of a constant vector is $\vec{0}$,

$$\vec{0} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix} + \begin{pmatrix} -aB_{\text{eq}} \\ -bA_{\text{eq}} \end{pmatrix}$$

$$\vec{0} = \begin{pmatrix} aF \\ bE \end{pmatrix} + \begin{pmatrix} -aB_{\text{eq}} \\ -bA_{\text{eq}} \end{pmatrix}$$

This yields the system

$$\begin{cases} a(F - B_{\text{eq}}) = 0 \\ b(E - A_{\text{eq}}) = 5 \end{cases} \implies \begin{matrix} F = B_{\text{eq}} \\ E = A_{\text{eq}} \end{matrix}$$

so

$$\vec{X}_p = \begin{pmatrix} A_{\text{eq}} \\ B_{\text{eq}} \end{pmatrix}$$

This makes sense because if both reservoirs are at equilibrium, then the differential equation is trivially satisfied because there is no change in the system.

The general solution is therefore

$$\vec{X} = \vec{X}_c + \vec{X}_p$$

$$\vec{X}(t) = c_1 \begin{pmatrix} \beta \\ 1 \end{pmatrix} e^{\alpha t} + c_2 \begin{pmatrix} -\beta \\ 1 \end{pmatrix} e^{-\alpha t} + \begin{pmatrix} A_{\text{eq}} \\ B_{\text{eq}} \end{pmatrix}$$

Using the initial conditions to solve for c_1 and c_2,

$$\vec{X}(0) = c_1 \begin{pmatrix} \beta \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -\beta \\ 1 \end{pmatrix} + \begin{pmatrix} A_{\text{eq}} \\ B_{\text{eq}} \end{pmatrix} = \begin{pmatrix} \beta c_1 - \beta c_2 + A_{\text{eq}} \\ c_1 + c_2 + B_{\text{eq}} \end{pmatrix}$$

$$\vec{X}(0) = \begin{pmatrix} \beta c_1 - \beta c_2 + A_{\text{eq}} \\ c_1 + c_2 + B_{\text{eq}} \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}$$

Subtracting the equilibrium amounts, let's let $A_d = A_0 - A_{\text{eq}}$ and $B_d = B_0 - B_{\text{eq}}$.

$$\begin{cases} \beta c_1 - \beta c_2 = A_d \\ c_1 + \quad c_2 = B_d \end{cases}$$

Multiplying the second row by β and adding the two equations to cancel c_2,

$$\begin{aligned} 2\beta c_1 &= A_d + \beta B_d \\ c_1 &= \frac{A_d + \beta B_d}{2\beta} \end{aligned}$$

Solving for c_2,

$$\begin{aligned} c_2 &= B_d - c_1 \\ &= \frac{2\beta B_d}{2\beta} - \frac{A_d + \beta B_d}{2\beta} = \frac{-A_d + \beta B_d}{2\beta} \end{aligned}$$

At last, the final solution is

$$\vec{X}(t) = \left(\frac{A_d + \beta B_d}{2\beta}\right) \begin{pmatrix} \beta \\ 1 \end{pmatrix} e^{\alpha t} + \left(\frac{-A_d + \beta B_d}{2\beta}\right) \begin{pmatrix} -\beta \\ 1 \end{pmatrix} e^{-\alpha t} + \begin{pmatrix} A_{\text{eq}} \\ B_{\text{eq}} \end{pmatrix}$$

Explicitly writing out A and B,

$$\begin{aligned} A(t) &= \left\{\frac{A_d + \beta B_d}{2}\right\} e^{\alpha t} + \left\{\frac{A_d - \beta B_d}{2}\right\} e^{-\alpha t} + A_{\text{eq}} \\ B(t) &= \left\{\frac{\beta B_d + A_d}{2\beta}\right\} e^{\alpha t} + \left\{\frac{\beta B_d - A_d}{2\beta}\right\} e^{-\alpha t} + B_{\text{eq}} \end{aligned}$$

The book actually has a typo for A! They have $A_0 - \beta B_0$ as the numerator for both fractions, when it should be $A_0 + \beta B_0$ for the first fraction (what I call A_d they call A_0).

> The solution to these two coupled differential equations can be shown to be
>
> $$\begin{aligned} A(t) - A_{\text{eq}} = &\left\{\frac{(A_0 - \beta B_0)}{2}\right\} \exp(\alpha t) \\ &+ \left\{\frac{(A_0 - \beta B_0)}{2}\right\} \exp(-\alpha t) \end{aligned}$$
>
> Here, A_0 and B_0 are the amounts that A and B are disturbed from their equilibrium values at the initiation of the disturbance, and $\alpha = \sqrt{ab}$ and $\beta = \sqrt{\frac{a}{b}}$. The second term on the right-hand side has a negative exponent and thus decays with time, but the first term has a positive exponent and thus will increase without limit if α is a real number. Thus, if the product ab is positive, as it must be for a positive feedback loop, the system is clearly unstable.

We continue our analysis, now on the case where $ab < 0$, that is, α is imaginary and the system represents a negative feedback loop. Recall the eigenvalues of the matrix were $\lambda = \pm\sqrt{ab}$. If $ab < 0$, then let $\alpha = \sqrt{-ab}$ to make α real again, making $\lambda = \pm i\alpha$.

For $\lambda = i\alpha$, we find its associated (complex) eigenvector:

$$\begin{pmatrix} -i\alpha & a \\ b & -i\alpha \end{pmatrix} \vec{K} = \vec{0} \to -i\alpha k_1 + ak_2 = 0 \to k_1 = \frac{ak_2}{i\alpha} = -i\frac{a}{\alpha}k_2$$

where we multiply by i on top and bottom to move i to the numerator. We introduce β again, but we have to be very careful how we define β in a complex space. Without loss of generality, let $a > 0$ and therefore $b < 0$. $\beta = \sqrt{-\frac{a}{b}}$, and $\frac{a}{\alpha} = \frac{\sqrt{a}\sqrt{a}}{\sqrt{a}\sqrt{-b}} = \beta$. Since we assume a is positive, we can decompose it. Finally, $k_1 = -i\beta k_2$. Taking $k_2 = 1$,

$$\vec{K} = \begin{pmatrix} -i\beta \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -\beta \\ 0 \end{pmatrix}$$

Using the technique proved in the appendix, section 4.1, we don't need to find the other eigenvector and can instead directly compute the two solutions from the real and imaginary parts of the eigenvector.

$$\vec{X}_1 = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos\alpha t - \begin{pmatrix} -\beta \\ 0 \end{pmatrix} \sin\alpha t \right] e^{0t} = \begin{pmatrix} \beta\sin\alpha t \\ \cos\alpha t \end{pmatrix}$$

$$\vec{X}_2 = \left[\begin{pmatrix} -\beta \\ 0 \end{pmatrix} \cos\alpha t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin\alpha t \right] e^{0t} = \begin{pmatrix} -\beta\cos\alpha t \\ \sin\alpha t \end{pmatrix}$$

The general solution is then

$$\vec{X} = c_1 \begin{pmatrix} \beta\sin\alpha t \\ \cos\alpha t \end{pmatrix} + c_2 \begin{pmatrix} -\beta\cos\alpha t \\ \sin\alpha t \end{pmatrix} + \begin{pmatrix} A_{\text{eq}} \\ B_{\text{eq}} \end{pmatrix}$$

where we make use of the same particular solution as the positive feedback case. Solving for the constants by making use of the initial conditions,

$$\vec{X}(0) = \begin{pmatrix} -\beta c_2 \\ c_1 \end{pmatrix} + \begin{pmatrix} A_{\text{eq}} \\ B_{\text{eq}} \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}$$

Moving the equilibrium sizes to the other side again,

$$\begin{cases} -\beta c_2 = A_d \\ c_1 = B_d \end{cases} \implies \begin{matrix} c_1 = B_d \\ c_2 = -\frac{A_d}{\beta} \end{matrix}$$

Plugging back into $\vec{X}$,

$$\vec{X} = B_d \begin{pmatrix} \beta\sin\alpha t \\ \cos\alpha t \end{pmatrix} - \frac{A_d}{\beta} \begin{pmatrix} -\beta\cos\alpha t \\ \sin\alpha t \end{pmatrix} + \begin{pmatrix} A_{\text{eq}} \\ B_{\text{eq}} \end{pmatrix}$$

$$A(t) = A_d \cos\alpha t + \beta B_d \sin\alpha t + A_{\text{eq}}$$

$$B(t) = B_d \cos\alpha t - \frac{A_d}{\beta} \sin\alpha t + B_{\text{eq}}$$

Because both $A(t)$ and $B(t)$ are a linear combination of sines and cosines, they can be written as a single sine function with the same frequency of α and a certain fixed magnitude and phase. Thus, the magnitudes of A and B are bounded above by the magnitude of that wave plus A_{eq} or B_{eq}, and below by the negative magnitude of the wave plus the same equilibrium size. For more details, consult section 4.5 of the appendix.

We finish with the last statement on the page.

> When a and b have opposite signs, though, as they do in a negative feedback loop, then the product ab is negative. The square root of a negative number is imaginary, so α is no longer a real number. In such a case, the system becomes a sinusoidal oscillator. The solution is always bounded, however, thus demonstrating that negative feedback loops are stable.

Our job is not quite done, however. Notice how $\beta = \sqrt{\frac{a}{b}}$. Clearly, this poses a problem if $b = 0$, and by symmetry, $a = 0$ is also bad—these cases are "degenerate".

If $a = 0$ and $b = 0$, the reader can confirm for themselves that both the positive and negative feedback loop solutions "do the right thing" and reduce to two constant solutions $A(t) = A_0$ and $B(t) = B_0$ which trivially satisfy the differential equations.

If exactly one of a or b is 0, assume $b = 0$ and therefore $a \neq 0$ without loss of generality. We could proceed with eigenvalue analysis as usual, discovering that $\lambda = 0$ is the only eigenvalue, and is "defective" in the sense that it has multiplicity 2 but only one associated eigenvector. We could handle this using a polynomial form, successively generating solutions via this sole eigenvector. See section 4.3 of the appendix for details. However, the system can be much more easily solved with ad-hoc analysis. Since $b = 0$, $\frac{dB}{dt} = 0$ so $B(t)$ is a constant function, whose value must be B_0 to satisfy the initial condition. $\frac{dA}{dt}$ then equals $a(B - B_{\text{eq}}) = aB_d$. Integrating on both sides, $A(t) = aB_d t + C$, and C must be A_0 to satisfy the initial condition. So $A(t) = aB_d t + A_0$ and $B(t) = B_0$, which means A grows without bound, but much slower than an exponential function.

3 Conclusion

We have rigorously computed the solution to the differential equation, correcting the mistakes of the geosystems textbook and extending to the complex and degenerate cases. Positive feedback loops happen when both couplings are positive or both are negative, while negative feedback loops happen when one is positive and the other is negative. We have seen the mathematical result of a positive feedback loop; the amount of material grows exponentially as both feed into each other. In a negative feedback loop, in contrast, the system oscillates perpetually, but is always stable, as its deviation from equilibrium is bounded. Future work can be done in extending this intuition to multiple couplings, where positive feedback loops happen if the number of positive coupling is even. Negative feedback loops, by process of elimination, must happen when the number is odd. Work can also be done in more complicated systems—for example, when the sign of a coupling is itself dependent on the state of the system, so an initially negative feedback loop can turn into a positive feedback loop with an external forcing (refer to the Daisyworld lab). These systems often cannot be solved analytically, necessitating a numerical approach.

References

[1] L. Kump, J. Kasting, and R. Crane, *The Earth System*, 3rd ed. 2010.

4 Appendix

4.1 Conjugate Eigenvalues and Eigenvectors

Let $\lambda = \alpha + i\beta$ be a complex eigenvalue of the coefficient matrix A in the homogeneous system $\vec{X}' = A\vec{X}$ and let $\vec{K}$ be a (complex) eigenvector associated with λ. Also let $\vec{u_1} = \mathrm{Re}(\vec{K})$ and $\vec{u_2} = \mathrm{Im}(\vec{K})$ such that $\vec{K} = \vec{u_1} + i\vec{u_2}$.

Also let $\overline{\lambda} = \alpha - i\beta$ be the ***conjugate*** eigenvalue of λ. We first prove that the eigenvector of $\overline{\lambda}$ is the conjugate of $\vec{K}$, or $\overline{\vec{K}} = \vec{u_1} - i\vec{u_2}$.

Because $\vec{K}$ is the eigenvector of λ by definition,

$$A\vec{K} = \lambda\vec{K}$$

Expanding on the left side,

$$\begin{aligned} A\vec{K} &= A(\vec{u_1} + i\vec{u_2}) \\ &= A\vec{u_1} + iA\vec{u_2} \end{aligned}$$

Expanding on the right side,

$$\begin{aligned} \lambda\vec{K} &= \lambda(\vec{u_1} + i\vec{u_2}) \\ &= (\alpha + i\beta)(\vec{u_1} + i\vec{u_2}) \\ &= (\alpha\vec{u_1} - \beta\vec{u_2}) + i(\beta\vec{u_1} + \alpha\vec{u_2}) \end{aligned}$$

Setting real and imaginary parts equal,

$$A\vec{u_1} = \alpha\vec{u_1} - \beta\vec{u_2} \text{ and } A\vec{u_2} = \beta\vec{u_1} + \alpha\vec{u_2}$$

We now compute $\overline{\lambda}\overline{\vec{K}}$ which should be equal to $A\overline{\vec{K}}$ if $\overline{\vec{K}}$ is an eigenvector of $\overline{\lambda}$.

$$\begin{aligned} \overline{\lambda}\overline{\vec{K}} &= (\alpha - i\beta)(\vec{u_1} - i\vec{u_2}) \\ &= (\alpha\vec{u_1} - \beta\vec{u_2}) + i(-\beta\vec{u_1} - \alpha\vec{u_2}) \end{aligned}$$

Using $A\vec{u_1}$ and $A\vec{u_2}$ computed earlier,

$$\begin{aligned} &= A\vec{u_1} - iA\vec{u_2} \\ &= A(\vec{u_1} - i\vec{u_2}) \\ &= A\overline{\vec{K}} \quad \square \end{aligned}$$

4.2 Real Solutions

We have two eigenvalues λ and its conjugate $\overline{\lambda}$ and we have the two associated eigenvectors $\vec{K}$ and its conjugate $\overline{\vec{K}}$.

Thus, the two (complex) solutions are $\vec{K}e^{\lambda t}$ and $\overline{\vec{K}}e^{\overline{\lambda}t}$. We want to write these two solutions in terms of reals, which is possible because we are allowed to arbitrarily linearly combine the two solutions by the superposition principle.

We first explicitly write out each solution in terms of $\vec{u_1}$ and $\vec{u_2}$.

$$\vec{K}e^{\lambda t} = (\vec{u_1} + i\vec{u_2})e^{(\alpha+i\beta)t}$$
$$= [\vec{u_1}e^{i\beta t} + i\vec{u_2}e^{i\beta t}]e^{\alpha t}$$

Using Euler's formula $e^{ix} = \cos x + i \sin x$ (a proof appears in the appendix, section 4.4),

$$= [\vec{u_1}(\cos\beta t + i\sin\beta t) + i\vec{u_2}(\cos\beta t + i\sin\beta t)]e^{\alpha t}$$

Seperating real and imaginary parts,

$$= [(\vec{u_1}\cos\beta t - \vec{u_2}\sin\beta t) + i(\vec{u_1}\sin\beta t + \vec{u_2}\cos\beta t)]e^{\alpha t}$$

We now do the same for the other solution.

$$\overline{\vec{K}}e^{\bar{\lambda}t} = (\vec{u_1} - i\vec{u_2})e^{(\alpha-i\beta)t}$$
$$= [\vec{u_1}e^{-i\beta t} - i\vec{u_2}e^{-i\beta t}]e^{\alpha t}$$

Because $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$,

$$= [\vec{u_1}(\cos\beta t - i\sin\beta t) - i\vec{u_2}(\cos\beta t - i\sin\beta t)]e^{\alpha t}$$

Beware of the sign on the $\vec{u_2}\sin\beta t$ term!
The two negative signs from each conjugation cancel:

$$= [(\vec{u_1}\cos\beta t - \vec{u_2}\sin\beta t) - i(\vec{u_1}\sin\beta t + \vec{u_2}\cos\beta t)]e^{\alpha t}$$

To summarize,

$$\vec{K}e^{\lambda t} = [(\vec{u_1}\cos\beta t - \vec{u_2}\sin\beta t) + i(\vec{u_1}\sin\beta t + \vec{u_2}\cos\beta t)]e^{\alpha t}$$
$$\overline{\vec{K}}e^{\bar{\lambda}t} = [(\vec{u_1}\cos\beta t - \vec{u_2}\sin\beta t) - i(\vec{u_1}\sin\beta t + \vec{u_2}\cos\beta t)]e^{\alpha t}$$

We notice we can cancel the imaginary parts if we add them up:

$$\frac{1}{2}(\vec{K}e^{\lambda t} + \overline{\vec{K}}e^{\bar{\lambda}t}) = \frac{1}{2}[2(\vec{u_1}\cos\beta t - \vec{u_2}\sin\beta t)]e^{\alpha t}$$
$$\vec{X_1} = [\vec{u_1}\cos\beta t - \vec{u_2}\sin\beta t]e^{\alpha t}$$

We notice we can cancel the real parts if we take the difference, and to make it real we can simply multiply it by a factor of i.

$$\frac{-i}{2}(\vec{K}e^{\lambda t} - \overline{\vec{K}}e^{\bar{\lambda}t}) = \frac{-i}{2}[2i(\vec{u_1}\sin\beta t + \vec{u_2}\cos\beta t)]e^{\alpha t}$$
$$\vec{X_2} = [\vec{u_2}\cos\beta t + \vec{u_1}\sin\beta t]e^{\alpha t} \quad \square$$

4.3 Defective Eigenvalues

Let λ be an defective eigenvalue with multiplicity m, with a single eigenvector by definition. Then each solution is of the form:

$$\vec{X}_m = \vec{K}_1 \frac{t^{m-1}}{(m-1)!} e^{\lambda t} + \vec{K}_2 \frac{t^{m-2}}{(m-2)!} e^{\lambda t} + \cdots + \vec{K}_m e^{\lambda t}$$

Proof. Inductive sketch. For $\vec{X}_m$ to be a valid solution, it must satisfy $\vec{X}'_m = A\vec{X}$.

$$\begin{aligned}\vec{X}'_m &= \vec{K}_1 \frac{t^{m-2}}{(m-2)!} e^{\lambda t} + \vec{K}_2 \frac{t^{m-3}}{(m-3)!} e^{\lambda t} + \cdots + \vec{0} \\ &+ \vec{K}_1 \frac{t^{m-1}}{(m-1)!} \lambda e^{\lambda t} + \vec{K}_2 \frac{t^{m-2}}{(m-2)!} \lambda e^{\lambda t} + \cdots + \vec{K}_m \lambda e^{\lambda t}\end{aligned}$$

By the inductive hypothesis, the top is $\vec{X}_{m-1}$:

$$= \vec{X}_{m-1} + \lambda \vec{X}_m$$

This must be equal to $A\vec{X}_m$, so

$$\vec{X}_{m-1} + \lambda \vec{X}_m = A\vec{X}_m$$

or, moving $\lambda \vec{X}_m$ to the right side,

$$(A - \lambda I)\vec{X}_m = \vec{X}_{m-1}$$

We need to show this system is solvable, and the easiest way to do that is to write it in terms of $\vec{K}_1 \ldots \vec{K}_m$ first and then reconstruct the solutions $\vec{X}_1 \ldots \vec{X}_m$.

$$\begin{aligned}(A - \lambda I)(\vec{K}_1 \frac{t^{m-1}}{(m-1)!} e^{\lambda t} + \vec{K}_2 \frac{t^{m-2}}{(m-2)!} e^{\lambda t} + \vec{K}_3 \frac{t^{m-3}}{(m-3)!} e^{\lambda t} + \cdots + \vec{K}_m e^{\lambda t}) = \\ \vec{K}_1 \frac{t^{m-2}}{(m-2)!} e^{\lambda t} + \vec{K}_2 \frac{t^{m-3}}{(m-3)!} e^{\lambda t} + \cdots + \vec{K}_{m-1} e^{\lambda t})\end{aligned}$$

Because $\vec{X}_m$ is one degree higher than $\vec{X}_{m-1}$, $\vec{K}_1$ has no correspondence, so we have

$$\begin{aligned}(A - \lambda I)\vec{K}_1 &= \vec{0} \\ (A - \lambda I)\vec{K}_2 &= \vec{K}_1 \\ &\cdots \\ (A - \lambda I)\vec{K}_m &= \vec{K}_{m-1}\end{aligned}$$

By induction, we can assume $\vec{K}_1 \ldots \vec{K}_{m-1}$ exist because the solution $\vec{X}_{m-1}$ exists. We therefore just need to show that $(A - \lambda I)\vec{K}_m = \vec{K}_{m-1}$ has a solution, which is left as an exercise to the reader. □

4.3.1 Defective Eigenvalues in the Degenerate Case

Recall we are trying to solve the system where $a \neq 0$ and $b = 0$. We proceed with eigenvalue analysis, where the characteristic equation is:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & a \\ 0 & -\lambda \end{vmatrix} = \lambda^2 - 0 = 0 \implies \lambda = 0$$

We proceed to find the sole eigenvector for the sole eigenvalue.

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \vec{K} = \vec{0} \to ak_2 = 0 \implies k_2 = 0, k_1 \text{ free}$$

Taking $k_1 = 1$,

$$\vec{K} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ so } \vec{X}_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{0t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We now solve for $\vec{P}$, which satisfies $(A - \lambda I)\vec{P} = \vec{K}$:

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \vec{P} = \vec{K}$$

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to ap_2 = 1 \implies p_2 = \frac{1}{a}, p_1 \text{ free}$$

Taking $p_2 = 0$ for simplicity,

$$\vec{P} = \begin{pmatrix} 0 \\ \frac{1}{a} \end{pmatrix} \text{ so } \vec{X}_2(t) = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ \frac{1}{a} \end{pmatrix} \right] e^{0t} = \begin{pmatrix} t \\ \frac{1}{a} \end{pmatrix}$$

The general solution is therefore

$$\vec{X} = c_1 \vec{X}_1 + c_2 \vec{X}_2 + \vec{X}_p$$
$$\vec{X} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t \\ \frac{1}{a} \end{pmatrix} + \begin{pmatrix} A_{\text{eq}} \\ B_{\text{eq}} \end{pmatrix}$$

where we used the standard particular solution. Making use of the initial conditions,

$$\vec{X}(0) = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{a} c_2 \end{pmatrix} + \begin{pmatrix} A_{\text{eq}} \\ B_{\text{eq}} \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}$$

$$\begin{cases} c_1 = A_d \\ \frac{1}{a} c_2 = B_d \end{cases} \implies \begin{array}{l} c_1 = A_d \\ c_2 = aB_d \end{array}$$

Plugging back into $\vec{X}$,

$$\vec{X} = A_d \begin{pmatrix} 1 \\ 0 \end{pmatrix} + aB_d \begin{pmatrix} t \\ \frac{1}{a} \end{pmatrix} + \begin{pmatrix} A_{\text{eq}} \\ B_{\text{eq}} \end{pmatrix}$$
$$A(t) = aB_d t + A_0$$
$$B(t) = B_0$$

Luckily, this agrees with our ad-hoc analysis. However, this method for this particular set of equations is clearly much more tedious than simply solving with separation.

4.4 Euler's Formula

Theorem 4.1. *$e^{ix} = \cos x + i \sin x$, i.e. Euler's formula*

Proof. We have the initial value problem (IVP)

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

Picard's existence and uniqueness theorem says that if f and $\frac{\partial f}{\partial y}$ are continuous functions on some rectangle R that contains (x_0, y_0), then the IVP has an unique solution on some interval I whose bounds are the regions where the hypotheses hold.

In our particular case, we have $f(x, y) = iy$ and $y(0) = 1$, so $\frac{\partial f}{\partial y} = i$. By Picard's theorem, the IVP has a unique solution on the interval I where y is continuous and $\frac{\partial f}{\partial y}$ is continuous. i is continuous everywhere, so the IVP will have an unique solution wherever y is continuous.

Because this differential equation is separable, we can directly solve for y.

$$\begin{aligned}
\frac{dy}{dx} &= iy \\
\int \frac{1}{iy} dy &= \int dx \\
\frac{1}{i} \ln |iy| &= x + C \\
\ln |iy| &= ix + C \\
iy &= Ce^{ix} \\
y &= Ce^{ix}
\end{aligned}$$

Taking into account the initial condition, $y(0) = 1 = Ce^0 = C$. So $y = e^{ix}$, which is continuous on $\mathbb{R}$. Picard's theorem therefore guarantees the uniqueness of this solution. However, note that $\cos x + i \sin x$ is also a solution to the IVP. First, it fulfills the initial condition since $y(0) = \cos 0 + i \sin 0 = 1$. Second, it fulfills the differential equation:

$$\begin{aligned}
\frac{dy}{dx} &= -\sin x + i \cos x \\
&= i^2 \sin x + i \cos x && \text{Definition of } i \\
&= i(\cos x + i \sin x) \\
&= iy
\end{aligned}$$

Since e^{ix} is the unique solution to the IVP on $\mathbb{R}$, $e^{ix} = \cos x + i \sin x$. ☐

4.5 Linear Combination of Sines and Cosines

Suppose we have a function in the form $f(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$. Can rewrite f to be of the form $A\sin(\omega t + \delta)$, i.e. as a sin function with the same frequency, but with a certain magnitude and phase? Using the identity $\sin(x+y) = \sin x \cos y + \cos x \sin y$,

$$A\sin(\omega t + \delta) = A\sin(\omega t)\cos\delta + A\cos(\omega t)\sin\delta = (A\sin\delta)\cos(\omega t) + (A\cos\delta)\sin(\omega t)$$

From this $c_1 = A\sin\delta$ and $c_2 = A\cos\delta$, so

$$A^2\sin^2\delta + A^2\cos^2\delta = c_1^2 + c_2^2 \implies A = \sqrt{c_1^2 + c_2^2}$$

Finally,

$$c_1 = A\sin\delta \to \sin\delta = \frac{c_1}{A} \implies \delta = \arcsin\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\right) = \arctan\left(\frac{c_1}{c_2}\right)$$

where the last step comes from the right triangle with opposite side c_1, adjacent side c_2, and hypotenuse $\sqrt{c_1^2+c_2^2}$. We can therefore write $f(t) = \sqrt{c_1^2+c_2^2}\sin\left(\omega t + \arctan\left(\frac{c_1}{c_2}\right)\right)$.

5 About the Author

The author's interest in geosystems and mathematics began at an early age, fascinated by the stars in the sky and the beauty of math. She currently takes geoscience at Naoetsu Private High School, and plans to study astronomy in college, aiming towards the creation of a complete space map, as well as solving the current mass asymmetries unexplainable by current models of galaxy formation. In her free time, the author likes going to the planetarium and going on dates, preferably both at the same time.

Figure 1: A picture of the author