Case Study of the Discord Mudae System

Stephen Huan

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1 Introduction

Discord is a popular messaging platform whose API is very useful, allowing many users to create "bots". These bots automate interactions with Discord users through text and other means to perform a specific function. Among these bots is Mudae, which allows the user to play a simple game. One types in the `$wa` command to "roll", which instructs Mudae to show a random character from television shows. One can then "claim" a character, which means they now have possession of the character. Note that one can roll 10 times per hour, but only claim once per 3 hours. Thus, there are 30 rolls per claim, and it is impossible to know *a prori* when to stop rolling and settle on a character.

Each character has a ***kakera value***, which is an objective determination of value irrespective of the emotional value one might attach to their favorite characters. Thus, our discussion centers around maximizing kakera. Suppose one rolls a 100 kakera value character. Should they stop rolling and claim this character, or continue rolling in hope of getting a higher value later? To answer this central question, we must use the methods of probability theory to analyze the kakera distribution. [1]

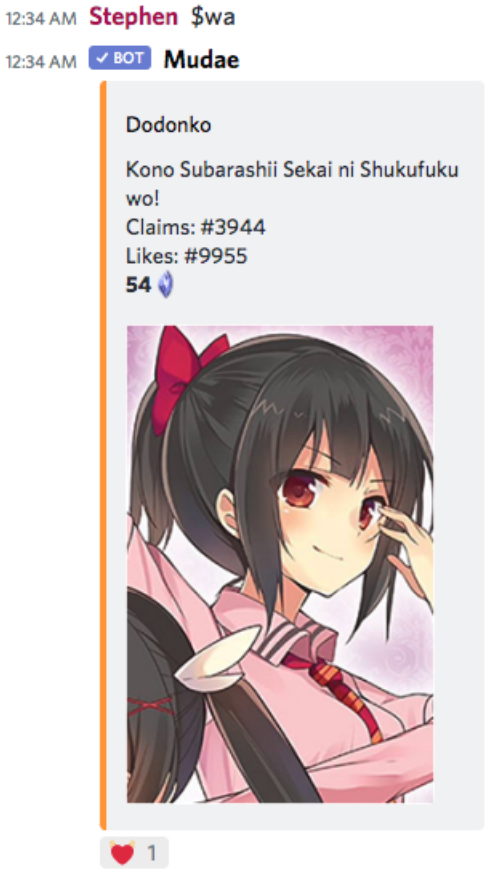


Figure 1: An example roll. The character name (Dodonko) is at the top, and the show they are from under it (Kono-Suba). The number left of the crystal icon indicates the kakera value, in this case 54. One clicks the heart icon under the Mudae modal to claim the character.

[1] This problem is very similar to the secretary problem, described in the appendix, subsection 9.1.

2 Probability Theory

If you are already familiar with the basic concepts of probability theory, e.g. random variables, PMFs, CMFs, expected value, and variance, feel free to skip to the next section.

2.1 Basic Definitions

Definition 2.1. A *random variable* (r.v.) is formally a function assigning a value to an outcome, but can more intuitively be thought of as a "dispenser" of random values. We will denote random variables as a single uppercase letter, e.g. X or Z.

A random variable is *continuous* if it dispenses infinite possible values (e.g. a random number between 0 and 1) and *discrete* if it dispenses a finite number of possible values (e.g. a coin). We are dealing with a discrete space—the kakera values are integer and vary between 32 and 1239, so we will focus on discrete random variables.

Definition 2.2. A *probability mass function* (PMF) gives the probability that a random variable dispenses a value. We will denote a PMF associated with the random variable X as $f_X(x)$, which gives the probability of the event x happening, $p(X = x)$.

Definition 2.3. The *cumulative mass function* (CMF) gives the probability that a random variable dispenses a value *less than or equal* to a given value. We will denote a CMF associated with the random variable X as $F_X(x)$, which gives $p(X \leq x)$.

Definition 2.4. The *expected value* is what we "expect" a random variable to dispense over many samples, or the average value. One way to compute this is:

$$\mathrm{E}[X] = \sum_{x \in X} xp(x)$$

or each value times the probability it occurs. The strong law of large numbers gives formal proof for why sampling a random variable and averaging the results converges to the expected value, but is beyond the scope of this paper. We also denote the expected value as μ, like an average.

Theorem 2.1. $\mathrm{E}[f(X)] = \sum_{x \in X} f(x)p(x)$*, i.e. the expected value of a transformation of a discrete random variable.*

Proof. Suppose f is a one-to-one or *bijective* mapping. Intuitively, $f(x)$ can only be dispensed in the transformed r.v. if x was dispensed in the original r.v., so the probability that $f(x)$ occurs is the probability that x occurs. Its value is still $f(x)$, so the sum $f(x)p(f(X) = f(x))$ becomes $f(x)p(x)$. Now we see what happens if f is not bijective, or if different values are mapped to the same value after the transformation. Then the probability of $f(x)$ occurring is the probability that any of $x_1, x_2, \ldots, x_n$ occur, if each $f(x_i) = f(x)$. Since each distinct x_i forms non-overlapping cases, the probability $f(x)$ occurs is the sum of their probabilities. Therefore $f(x)$ "contributes" $f(x)p(f(X)) = f(x)(p(x_1) + p(x_2) + \cdots + p(x_n)) = f(x)p(x_1) + f(x)p(x_2) + \cdots + f(x)p(x_n)$ to the expected value, which is accounted for in the sum of $f(x)p(x)$ over each value of X. □

Corollary 2.1.1. *The linearity of expectation, i.e.*

1. $\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y]$ *for random variables* X, Y
2. $\mathrm{E}[cX] = c\,\mathrm{E}[X]$ *for* $c \in \mathbb{R}$

Proof.

$$\begin{aligned}\mathrm{E}[X+Y] &= \sum_{x\in X}\sum_{y\in Y}(x+y)p(X=x,Y=y)\\ &= \sum_{x\in X}\sum_{y\in Y}xp(X=x,Y=y) + \sum_{x\in X}\sum_{y\in Y}yp(X=x,Y=y)\end{aligned}$$

Since interchanging the order of the summations just changes the order of addition, we can interchange on the right term while factoring out the x and y terms:

$$\begin{aligned}&= \sum_{x\in X} x\underbrace{\sum_{y\in Y}p(X=x,Y=y)}_{p(X=x)} + \sum_{y\in Y} y\underbrace{\sum_{x\in X}p(X=x,Y=y)}_{p(Y=y)}\\ &= \sum_{x\in X}xp(x) + \sum_{y\in Y}yp(y)\\ &= \mathrm{E}[X] + \mathrm{E}[Y]\end{aligned}$$

We use the transformation $f(x) = cx$ with our previous theorem:

$$\begin{aligned}\mathrm{E}[cX] &= \sum_{x\in X}(cx)p(x)\\ &= c\sum_{x\in X}xp(x)\\ &= c\,\mathrm{E}[x]\end{aligned}$$

□

Definition 2.5. The *variance* is the expected squared deviation from the expected value. The larger the variance, the more "variable" the random variable is. By definition,

$$\mathrm{Var}[X] = \mathrm{E}[(X - \mathrm{E}[X])^2]$$

We also denote the variance as σ^2, since it is the square of the standard deviation σ.

Theorem 2.2. $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$*, a more convenient way to calculate variance.*

Proof.

$$\begin{aligned}\mathrm{Var}[X] &= \mathrm{E}[(X-\mathrm{E}[X])^2] && \text{By definition}\\ &= \mathrm{E}[X^2 - 2X\,\mathrm{E}[X] + \mathrm{E}[X]^2]\\ &= \mathrm{E}[X^2] - \mathrm{E}[2\,\mathrm{E}[X]X] + \mathrm{E}[\mathrm{E}[X]^2] && \text{Linearity of expected value}\\ &= \mathrm{E}[X^2] - (2\,\mathrm{E}[X])\,\mathrm{E}[X] + \mathrm{E}[X]^2 && \text{Expectation of a constant is constant}\\ &= \mathrm{E}[X^2] - \mathrm{E}[X]^2\end{aligned}$$

□

2.2 Computational Representation

We now represent these concepts with a computer. Many of the concepts will be language-agnostic, but for practical purposes I will describe everything in terms of the high-level programming language Python. All code can be found at my GitHub repository.

2.2.1 Random Variables and the PMF

We will represent a random variable as two *lists*, one holding the values and the other their corresponding probabilities. We make sure the value list is sorted, and for a given index i its value is $X[i]$ and its PMF is $p[i]$ for the tuple of lists (X, p).

an example random variable

```
X = [   1,    5,  10,   15,   16,   20,  30,   35,   50,  100]
p = [0.05, 0.08, 0.1, 0.12, 0.17, 0.22, 0.1,  0.1, 0.05, 0.01]
```

We now show the PMF function, as opposed to the PMF list. The PMF list maps index to probability, but we often want value to probability. In order to do this efficiently, we can pre-compute a dictionary mapping value to index and then simply lookup the index in the PMF list, a total query time of $O(1)$. If the value isn't in the dictionary, then it does not occur in X and thus has a probability of 0.

pmf function

```
D = {x: i for i, x in enumerate(X)} # value to index

def pmf(p: list, u: float) -> float:
    """ Finds the pmf at a value in or not in the underlying r.v. """
    return p[D[u]] if u in D else 0
```

2.2.2 The CMF

With the PMF complete, we analyze the CMF. The CMF of an index is simply the sum of the probabilities up to and including that index. Similar to the PMF, we can make our queries more efficient if we do some pre-computation.

Consider the problem of finding the area under the curve for a function $f(x)$ between the points $x = a$ and $x = b$. We do not need to integrate again for two different points! We can simply do one *indefinite* integral, $F(x) = \int f(x)dx$, and use this function F to compute the area: $F(b) - F(a)$. One way to think about this is to imagine the function $F(x)$ as the area between a hypothetical point $x = c$ to x. This point might not actually exist, but as an example take the line $f(x) = x$. Its integral is $F(x) = \frac{1}{2}x^2$, and because $F(0) = 0$ we can imagine $F(x)$ as giving the area between $x = 0$ and x (since $\int_0^x f(t)dt = F(x) - F(0) = F(x)$). Thus, if we want to find the area under curve between $x = a$ and $x = b$, then $F(b)$ gives the area from 0 to b, and subtracting $F(a)$ will remove the area from 0 to a, leaving the area between a and b.

We can apply a similar logic to the discrete case. Instead of the integral being our accumulation function, we define the ***prefix sum***, a list of length $n+1$ that holds the sum of each prefix of the original list. If we want the prefix sum of the list p, then the first value is 0, the second $p[0]$, the third $p[0]+p[1]$, the fourth $p[0]+p[1]+p[2]$, and so on. In general, the ith index of the prefix sum gives `sum(p[:i])`, hence "prefix sum" (each element holds the sum of the ith prefix). The sum between two indexes i and j, inclusive on both ends, can then be computed in $O(1)$ by $F[j+1]-F[i]$. Note that this is very similar to the integral: $F(b)-F(a)$, hence the nickname "discrete integral".

Algorithm 1 Prefix sum of a list

```
def query(prefix: list, i: int, j: int) -> float:
    """ Finds the sum of a list between two indexes. """
    return prefix[j + 1] - prefix[i]

def prefix_sum(l: list) -> list:
    """ Returns the prefix sum of l. """
    prefix = [0]*(len(l) + 1)
    for i in range(len(l)):
        prefix[i + 1] = prefix[i] + l[i]
    return prefix
```

To find the CMF of a value x, we first find its index i with the dictionary `D` like the PMF. We then return $F[i+1]$, since we want to *include* the probability corresponding to x (CMF is less than *or equal*). Unlike the PMF, the CMF takes nonzero values for x not in the range of the random variable. In that case, we must find the largest index corresponding to a value less than or equal to x and then add 1 like the previous case. Equivalently, we can simply find the *smallest* index corresponding to a value strictly *greater* than x, and use that since the prefix sum will not include it. Since our random variable has its values in sorted order, we can do this search efficiently with binary search, through Python's bisect module. Thus, our final algorithm is $O(1)$ for values in the range and $O(\log n)$ for values not in the range.

cmf function

```
def cmf(F: list, u: float) -> float:
    """ Finds the cmf at a value in or not in the underlying r.v. """
    # use F if u is in the support set, otherwise binary search
    return F[D[u] + 1] if u in D else F[bisect.bisect(X, u)]
```

2.2.3 Expectation and Variance

Expectation and variance can be implemented from definition. For expectation, we include an transformation function whose default value is the identity function $f(x)=x$. Thus, $\mathrm{E}[f(x)] = \sum_{x\in X} f(x)p(x) =$ `sum(f(X[i])*p[i] for i in range(len(X)))`.

expectation

```
def E(p: list, rv: list=X, f=lambda x: x) -> float:
    """ Expected value of a pmf represented by a list. """
    return sum(f(rv[i])*p[i] for i in range(len(rv)))
```

For variance, we use the derived definition $\text{Var}[X] = \text{E}[X^2] - \text{E}[X]^2$.

variance

```
def Var(p: list, rv: list=X) -> float:
    """ Var[X] = E[(x - u)^2] = E[X^2] - E[x]^2. """
    return E(p, X, lambda x: x*x) - E(p, rv)**2
```

Finally, for standard deviation we simply take the square root of variance.

standard deviation

```
def std(p: list, rv: list=X) -> float:
    """ sigma^2 = Var[x] so sigma = std. dev. = sqrt(Var[X]). """
    return math.sqrt(Var(p, rv))
```

Why use standard deviation instead of variance? One intuitive answer is that the units for variance is squared, so taking the square root in the standard deviation maintains the same units as the original data.

2.2.4 Empirical Distributions and Testing Validity

In many situations, we do not know *a priori* how a discrete distribution behaves and must figure out the probabilities by doing a frequency count. For example, if we do 10^3 trials and the value 5 occurs 125 times, then $p(X = 5)$ is roughly $\frac{125}{1000} = \frac{1}{8}$. Of course, this estimate gets better the more trials one does (the variance lowers).

If we have a frequency count of each value, we can normalize it into a PMF by diving each value by its sum, making the sum of the overall list 1.

normalizing a list into a pmf

```
def norm(l: list, f: float=None) -> list:
    """ Normalizes a list into a pmf by dividing by its sum. """
    s = sum(l) if f is None else f
    return [x/s for x in l]
```

Similarly, for a CMF the largest value possible has a CMF of 1 since every value is less than it. Therefore, we can normalize a list into a CMF by dividing by its max.

normalizing a list into a cmf

```
def norm_cmf(l: list) -> list:
    """ Normalizes a list into a cmf by dividing by its max. """
    return norm(l, max(l))
```

These normalization methods cannot turn an arbitrary list into a PMF or a CMF, so we must test whether the resulting list is valid. How do we determine whether a given list p is a valid PMF? It must have two necessary and sufficient properties: each value is nonnegative and the sum of the list is 1. In practice, we will not include values with 0 probability and we cannot test whether `sum(p) == 1` directly because of numerical error, we instead determine whether the distance is within a certain tolerance to 1.

valid pmf

```
def nonneg(l: list) -> bool:
    """ Whether all the values in a list are positive. """
    return all(map(lambda x: x >= 0, l))

def diff(x: float, y: float, tol: float=EPSILON) -> bool:
    """ Whether two numbers are sufficiently close. """
    return abs(x - y) < tol

def pmf(l: list, tol: 10**-3) -> bool:
    """ Whether a list can be interpreted as a pmf. """
    return nonneg(l) and diff(sum(l), 1, tol)
```

Thus, a random variable (X, p) is valid if X is sorted and p is a valid PMF.

checking the validitiy of a random variable

```
assert len(X) == len(p), "values not the same length as pmf"
assert sorted(X) == X,   "support set must be sorted"
assert pmf(p),           "not a valid pmf"
```

To test whether a CMF is valid, we need it to fulfill three properties: each value must be nonnegative and monotonically increasing since the CMF is formed from the sum of the nonnegative PMF, and the last element must be 1 since the largest value is greater than or equal to all other values.

valid cmf

```
def cmf(l: list, tol: float=10**-3) -> bool:
    """ Whether a list can be interpreted as a cmf. """
    return nonneg(l) and l == sorted(l) and diff(l[-1], 1, tol)
```

2.2.5 Sampling from a Distribution

Our final tool in our growing toolbox is the ability to sample from a random variable. However, most programming languages only support a uniform random variable (Python has `random.random()`, which gives a random number between 0 and 1) and a few others. We must, therefore, find a way to convert a sample from a uniform random variable into an arbitrary one. Perhaps it is easier to show the code first, then explain how it works.

sampling from a random variable

```
def sample(p: list) -> float:
    """ Samples a value from a random variable. """
    r = random.random()
    i = cmf = 0
    while i < len(p) - 1:
        cmf += p[i]
        if r < cmf:
            break
        i += 1
    return i
```

Claim: This function has the same CMF as the underlying probability distribution. For a uniform random variable $X \sim [0, 1]$, the cumulative distribution function or the probability X is less than some value x is $p(X \leq x) = \int_0^x 1dt = x$. `sample` will output an index $\leq j$ if the sum of the probabilities up to j is greater than X, triggering the `break`. Flipping the inequality, if X is less than the sum then `sample` outputs an index $\leq j$:

$$p(\texttt{sample} \leq j) = p(X \leq \sum_{i=0}^{j} p[i]) = \sum_{i=0}^{j} p[i]$$

Recall that this is the prefix sum, the CMF of a discrete random variable. Thus, `sample` has the same CMF as underlying random variable, and because the PMF can be uniquely recovered from the CMF, it must have the same PMF. If it has the same PMF, then this is "sampling the probability distribution" represented by p by definition!

Another, more general, way to think about the same thing is to consider `sample` as finding the *inverse* of the CMF. This inverse is not too well-defined in a discrete setting; we can perhaps consider it as giving the smallest value x such that $F(x)$ is greater than X. In a continuous setting, this would be simply $F^{-1}(x)$, hence "inverse". The CMF of `sample` is then $p(\texttt{sample} \leq x)$ which is $p(F^{-1}(X) \leq x)$, or the inverse CMF applied to the random variable X. The key observation is that the CMF is monotonically increasing, which by definition means that if $a \leq b$, then $F(a) \leq F(b)$. We can therefore apply F on both sides of the inequality while preserving it because of monotonicity. The CMF of `sample` is then $p(X \leq F(x))$, which is the CDF of X, $F_X(F(x))$. Because X is a uniform random variable, this is simply $F(x)$ and we arrive at the conclusion `sample` has the same CMF as the underlying random variable again. This proof is more formal however, and clearly generalizes much easier to continuous random variables. For more information on the "inverse CDF technique" or the "CDF of CDFs" trick, see the Wikipedia article.

This inverse technique is more than just a mathematical trick; we will use it to derive a faster algorithm. Notice that the CMF as a prefix sum list is a mapping from index to probability value, and we want the inverse mapping from (uniform) probability value to index. This is just binary search again! We again find the *smallest* index i that is *larger* than our random sample, and use it to index the random variable list X to get a value

(not to be confused with the uniform random variable). We have to index X at $i-1$ because the prefix sum has 1 larger indexes (the index n in the prefix sum corresponds to the element $n-1$, and so on). This gives a $O(\log n)$ algorithm instead of $O(n)$.

Algorithm 2 Sampling from a random variable in $O(\log n)$

```
def sample(X: list, F: list) -> float:
    """ Samples a value from a random variable. """
    p = random.random()
    return X[bisect.bisect(F, p) - 1]
```

3 Mudae Models

3.1 Initial Model

We now resume our analysis of the original Mudae game. The first step is to get data, so we know what the kakera random variable is. This was done by Luke Thistlethwaite and Avik Rao, and generously provided to me. See Luke's repository here. Note that the data is dependent on the Discord server as well as the time of scraping (along with other, unknown, factors), so to exactly replicate my results you must use my data.

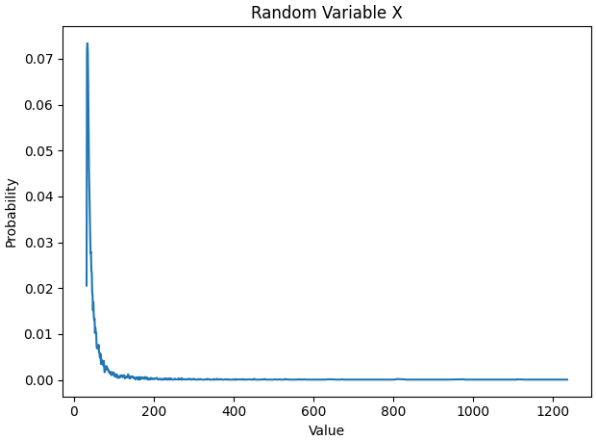


Figure 2: The kakera distribution. Large right skew—perfect for gamblers.

The data is of the form (X, p) as previously described. We now consider the optimal rolling strategy, answering the question posed in the introduction, section 1. Suppose we roll a character worth 100 kakera and we have 16 rolls left. Should we claim or continue rolling? The insight is that we can assume the value of the 16 rolls is going to be the *expected value* of those rolls, which is true over a large number of samples. If we therefore compute the expected value for 1 roll left, 2 rolls left, and so on until 30 rolls left, we can get cutoff values for each roll. If the expected value of 16 rolls is 120, then we should continue rolling. If the expected value is 80, then we should claim the 100 kakera character and stop rolling. We now see how to efficiently compute these expected values.

The overarching technique is known as "dynamic programming", covered in many Senior Computer Team lectures. We need not give such a formal description, however, it is simpler to describe the technique as "reducing the problem into smaller subproblems". Suppose we have 0 rolls left. Then the expected value is 0, since we can't gain kakera if we can't roll. If we have 1 roll left, then the expected value is just the expected value of the kakera random variable, which we call X. The interesting case comes in when we have 2 rolls left. Since we know the value of 1 roll left is $\mathrm{E}[X]$, then if we roll a character greater than the cutoff, we keep it, and if we roll less than the cutoff, we continue rolling and keep the $\mathrm{E}[X]$ value of 1 roll left. Thus, the expected value for 2 rolls is going to be a "capped" expected value sum where the value of x is the maximum between x and the previously computed value of having one less roll. If we call the random variable with r rolls left X_r, then we have a recurrence relation:

$$\mathrm{E}[X_r] = \mathrm{E}[\max(X, \mathrm{E}[X_{r-1}])] = \sum_{x \in X} \max(x, \mathrm{E}[X_{r-1}])p(x)$$

Of course, for any recurrence relation we need a base case, and we already found it: $\mathrm{E}[X_0] = 0$. It can be verified that the relation gives $\mathrm{E}[X_1] = \mathrm{E}[X]$ because we assume kakera values are necessarily positive.

We now use dynamic programming to compute these expected values quickly. If we used the recurrence relation naively, to compute $\mathrm{E}[X_r]$ we must do r steps down to $\mathrm{E}[X_0]$. However, note that in doing so we compute each $\mathrm{E}[X_0], \mathrm{E}[X_1], \ldots, \mathrm{E}[X_r]$. Thus, if we repeatedly query $\mathrm{E}[X_r]$ (as we do while rolling), then we can store each intermediate result and simply recall the stored value when the user requests it. This storage could be done with a list, but is also elegantly done with Python's `lru_cache` from the functools module, a technique called "memoization". We therefore do at most $r_{\max}$ iterations of the expected value summation, where $r_{\max} = 30$.

Algorithm 3 Capped expected value

```
def capped(X: list, p: list, u: float) -> float:
    """ Returns E[X], but with the values capped at a minimum of u. """
    return E(p, X, lambda x: max(x, u))

@lru_cache(maxsize=None)
def Er(X: list, p: list, r: int) -> float:
    """ E[X_r], where X_r is X with possibly r more samples. """
    # if r = 0, then we're out of samples and has a constant value of 0
    return 0 if r <= 0 else capped(X, p, Er(i, r - 1))
```

Finally, we show how to speedup the `capped` function. The observation is that since the X list is sorted, there is some index i such that every index less than i has a value less than or equal to $\mathrm{E}[X_{r-1}]$, and every index greater than or equal to i has a value greater than $\mathrm{E}[X_{r-1}]$. We can therefore partition X into two halves like in quicksort; in

the left half we use the value $\mathrm{E}[X_{r-1}]$ and in the right we use the original value:

$$\sum_{j=0}^{i-1} \mathrm{E}[X_{r-1}]p(x_j) + \sum_{j=i}^{n} x_j p(x_j)$$

$$= \mathrm{E}[X_{r-1}](p[0] + p[1] + \cdots + p[i-1]) + \sum_{j=i}^{n} x_j p(x_j)$$

Notice that $F[i] = p[0] + p[1] + \cdots + p[i-1]$ and that the right hand side is simply a suffix sum of the expected value. We can therefore also run a prefix sum on the expected value to efficiently calculate it, since the prefix sum gives the sum between any two indexes.

fast capped

```python
# expected value prefix sum
ev = prefix_sum([X[i]*p[i] for i in range(N)])

def capped(X: list, F: "cmf", prefix: list, u: float) -> float:
    """ Returns E[max(X, u)] in O(log n). """
    i = bisect.bisect(X, u)
    return u*F[i] + query(prefix, i, N - 1)
```

The exact distinction of less than or equal to will not be relevant, since if a value x is exactly equal to $\mathrm{E}[X_{r-1}]$, it does not matter whether it goes into the left or right side. Our algorithm is now $O(\log n)$ per query of `Er` (amortized), so $O(r_{\max} \log n)$ overall.

Armed with this algorithm, we can directly apply it to our data:

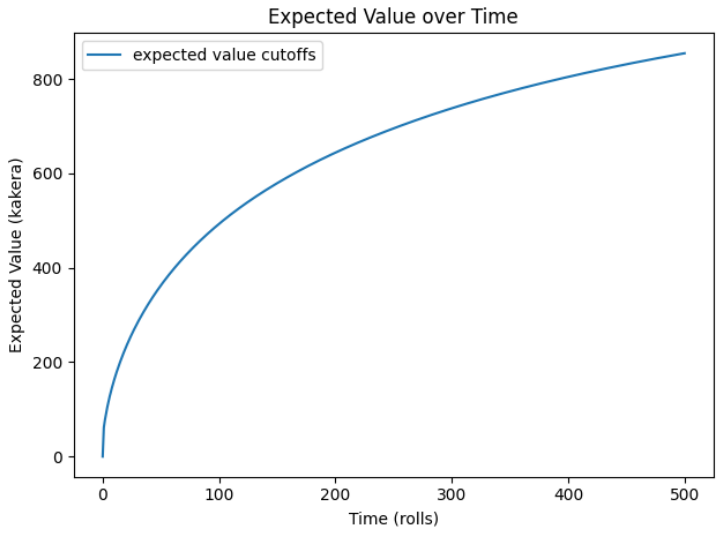


Figure 3: Expected value over increasing rolls.

As expected, the expected value increases sharply in the beginning and gradually levels off. It will asymptote at the maximum value, but takes many rolls to reach that state (since the higher kakera characters are so rare).

3.2 Batching

Satisfied with our initial model, we now successively refine it. The game I presented was a simplification; in reality one gets 10 rolls per hour, and is able to claim any character within about a minute. Thus, if someone rapidly runs `$wa`, they can take their pick of any of the 10 characters rolled. Assuming the goal is to maximize kakera, they obviously take the maximum kakera value of those characters.

3.2.1 Derivation of the Max of i.i.d. Random Variables

The probability-theoretic question is as follows: what is the distribution of the max of n independent and identically distributed (i.i.d.) random variables? As is common in probability, the easiest way to find the PMF of a discrete random variable is to first find the CDF of a continuous random variable.

Suppose the random variable X has CDF $F(x)$ and PDF $F'(x) = f(x)$, and that the random variable $Z = \max(X_1, X_2, \ldots, X_n)$ is the max of n samples of X. We wish to find the CDF of Z. By definition, the CDF is the probability the random variable is less than or equal to x. If at least one of the random variables was greater than x, then the max would be greater. Therefore for the max of the random variables to be less or equal to x, they all have to be less than or equal to x. Since each has a CDF of F and they are independent, the probability each is less than x is the product:

$$p(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x) = F(X_1)F(X_2)\ldots F(X_n) = F(X)^n$$

To find the PDF, we can simply differentiate with respect to x, using the chain rule:

$$g(x) = \frac{dF}{dx} = nF(X)^{n-1}f(x)$$

where $g(x)$ is the PDF of the random variable Z.

We now apply the continuous derivation to the discrete case. While it is true that $G[i] = F[i]^n$ where G is the CMF of Z and F is the CMF of X, we cannot differentiate. We instead use a "discrete derivative", or subtract adjacent indexes of G to recover the PMF g. Since $G[i] = g[0] + g[1] + \cdots + g[i-1]$ and $G[i+1] = g[0] + g[1] + \cdots + g[i]$, $g[i] = G[i+1] - G[i]$. Lastly, we index Z by its n, e.g. $Z_3 = \max(X_1, X_2, X_3)$. We therefore compute $Z_1, Z_2, \ldots, Z_B$ where $B = 10$ since we get 10 rolls per "batch". [2]

cmf of Z

```
Fzs = [[pow(v, b) for v in F] for b in range(B + 1)]
```

As derived, the PMF of Z_b is the adjacent differences of the CMF.

pmf of Z

```
def fz(z: float, b: int=B) -> float:
    """ pmf of Z. """
    return Fzs[b][D[z] + 1] - Fzs[b][D[z]]
```

[2] Yes, we can avoid the use of `pow` and shave a log factor by using Z_{B-1} to compute Z_B since $Z_B[i] = F[i]Z_{B-1}[i]$. Since $B = 10$ is small, I show the simpler method.

Note that while we use Python's builtin `pow` function, we could implement fast exponentiation ourselves. We want to compute x^e where x is a float and e is necessarily an integer. If we think of the exponent e as a binary number, we can write x^e as the product of x to the power of some powers of 2. The trick is that if we repeatedly square x, we get powers of 2, e.g. $x^2, x^4, x^8, \ldots$. Thus, we can get to the highest set bit (most significant bit) of e in $O(\log e)$, and by keeping track of the powers we used to get there we can therefore compute x^e in $O(\log e)$. There is no reason to use our function, however, since under the hood Python's `pow` uses this technique for integer powers.

Algorithm 4 Fast exponentiation

```
def exp(b: int, e: int) -> int:
    """ Compute b^e in O(log e). """
    rtn = 1
    while e > 0:
        # if bit on in the binary representation of the exponent
        if e & 1 == 1:
            rtn *= b
        e >>= 1
        b *= b
    return rtn
```

With the PMF of Z derived, we can visualize it with a graph:

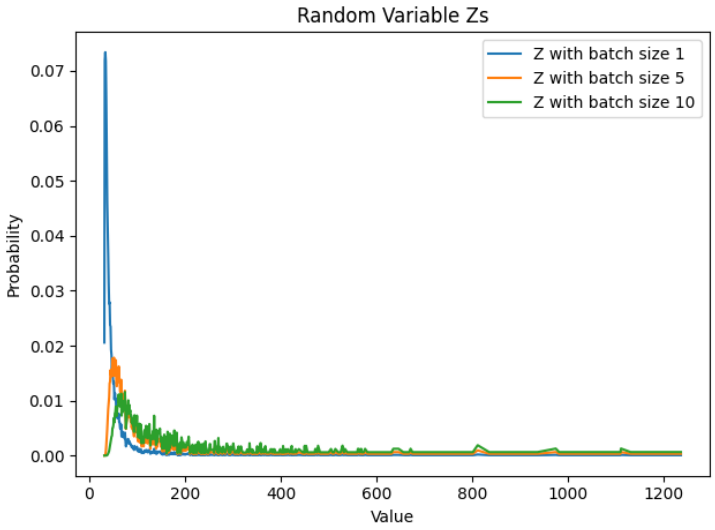


Figure 4: Random variables Z_5 and Z_{10} plotted against X

The effect of the max sharply decreases the peak of the PMF, instead spreading out the probability mass over the right skew. In general, the distribution is shifted to the right as expected; a larger batch size will monotonically increase kakera value.

3.2.2 Naive Implementation

We now apply the derived PMF and CMF of Z to the original problem of determining when to claim a character. Clearly, we can recycle our original capped expected value. We simply substitute (Z_B, p_{Z_B}) for (X, p) and change $r_{\max}$ to 3 since each "roll" (sample of the random variable) is equivalent to 10 rolls.

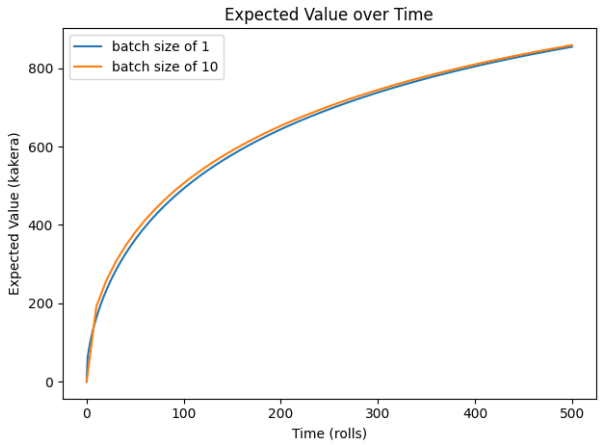


Figure 5: Batch size of 1 (previous model) versus batch size of 10.

As expected, the extra information provided by the ability to choose the largest value out of a batch of 10 rolls yields a slight advantage over immediately deciding whether or not to claim after a roll (which can be thought of as a batch size of 1). This advantage decreases over time as both curves asymptote.

There is one problem, however, and that is our new model only works for rolls that are a multiple of 10 (since we can only move in samples of Z_{10}). One notices in the graph that the orange starts *under* the blue, since it is at 0 for the first 10 rolls. This is a problem because people do not necessarily roll in perfect batches of 10; one might hypothetically be too slow or "sell" their rolls to other users after they have claimed, letting another person claim from their now useless rolls (a practice analyzed in subsection 3.5).

3.2.3 Fractional Batching

We now consider how to extend the previous "batched" model to any number of rolls. If the number of rolls is not a multiple of 10, we simply use $Z_{r\%10}$ instead of Z_{10}, where % is the *modulo* operator, which gives the integer remainder after dividing r by 10. If r is a multiple of 10, then $r\%10$ is 0 so we must explicitly default to Z_{10}.

Algorithm 5 Fractional batching model

```
@lru_cache(maxsize=None)
def Ef(r: int, b: int=B) -> float:
    """ E[X_r], where Z_r is extended to all r's. """
    n = r % b if r % b != 0 else b
    return 0 if r <= 0 else capped(Z, Fzs[n], evzs[n], Ef(r - n, b))
```

In terms of notation, we denote the random variable with r rolls left as X_r, and $\mathrm{E}[X_r]$ is now computed as shown above, instead of the previous immediate decision method. Using this more general function, we can correct Figure 5:

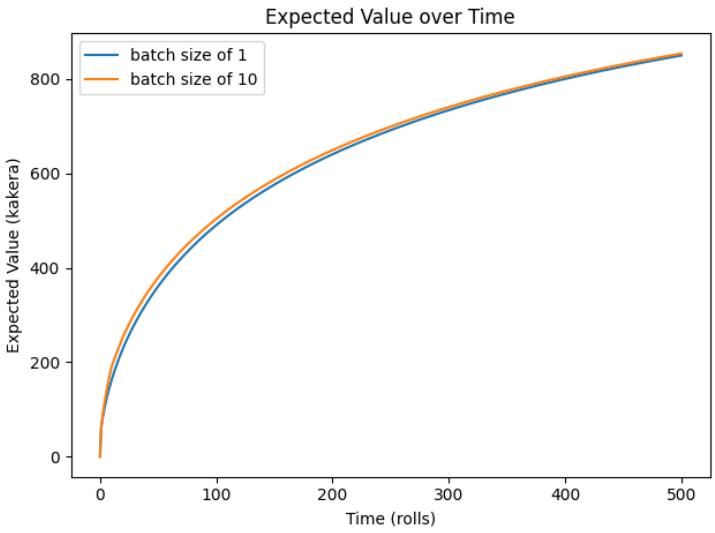


Figure 6: Batched model adjusted for fractional batches.

While we are getting a feel for the model, we can also experiment with changing the batch size while the number of rolls is fixed.

Figure 7: Expected value over increasing batch size, $r = 50$.

As batch size increases, expected value increases seemingly linearly. The flat part at the end is because the number of rolls is 50, so a batch size greater than 50 has no effect.

3.3 Putting it Together

Satisfied with our model, we now create an agent which implements a concrete rolling strategy according to our theoretical calculations. This will serve three purposes: it is easier to work with from the perspective of an end user since we are essentially creating an API, it will force us to somewhat simulate the system so we can use this model for other downstream purposes, and finally it will let us use code to test whether the math aligns with reality. Looking towards the future, we will implement one last model in subsection 3.4 that is *a posteriori*; its behavior depends in part on what one actually rolls. Thus, we need an agent to model such behavior.

Our agent will take R, the maximum number of rolls, and B, the batch size. Because it will frequently access the expected value cutoffs, we pre-compute these values and store them. Finally, we will maintain all instance variables in the `reset` function, which will allow us to reset the game without needing to create a new object.

init method

```
class Model():

    """ Models interactions with the kakera random variable. """

    def __init__(self, R: int=prob.R, B: int=prob.B) -> None:
        self.R, self.B, self.offset = R, B, R % B
        # precompute list of expected values for each roll
        self.E = [prob.Ef(r, B) for r in range(R + 1)]
        self.reset()
```

What attributes are specific to each game? We must know how many rolls we have left, r. If we are in the middle of a batch, we must know the kakera values of the past characters rolled which we will store in a list `l`. It is not strictly necessary but helpful to maintain the largest kakera value seen so far in a batch, b. Finally, to account for incomplete batches, we assume that an incomplete batch will be the first to finish, and then the rest of the batches are complete. Therefore we maintain the `size` variable, which tells us how many values the current batch will hold. It is initially `offset`, defined in `__init__` to be $R\%B$, and if `offset` is 0, it is the standard batch size B.

reset method

```
def reset(self) -> None:
    """ Reset to the inital model. """
    # number of rolls left, list of seen kakera values, best value
    self.r, self.l, self.b = self.R, [], float("-inf")
    self.size = self.offset if self.offset > 0 else self.B
```

The single method necessary for the operation of the model is `update`, which takes in a kakera value k that represents a roll. If the model decides to claim, it returns the index of the value in the current batch it wants to claim. Otherwise, it returns `None`.

We first determine whether or not the current batch has ended. If it has, we reset `l` to a empty list, we set b to $-\infty$, and we set `size` to B (since we assume the first batch is the one containing the one possible offset).

We then update the model's state with the new kakera value. We add k to `l`, and set b to $\max(b, k)$. We also decrement r since we just used a roll.

With the bookkeeping finished, we are finally ready to determine whether or not to claim this value (or any value in the current batch). As justified in section 3, we claim if our current value is larger than the expected value of the number of rolls left, which is r since we already decremented r. Thus, we compare b to $E[X_r]$. However, for true optimality we need another condition. It never makes sense to end a batch midway, since we can always finish the batch out and then make a decision. This seems to contradict our algorithm, but it will be resolved in subsection 3.5. In terms of the `update` function, however, all that is important is that we only claim when the current batch is over, when the length of `l` is `size`.

update method

```
def update(self, k: int) -> int:
    """ Returns an index if claiming, otherwise None. """
    # new batch, reset seen and b, the max of the samples we've seen
    if len(self.l) == self.size:
        self.l, self.b, self.size = [], -float("inf"), self.B
    # update variables with new information
    self.l.append(k)
    self.b = max(self.b, k)
    self.r -= 1

    # no need to stop early, wait until the batch is complete
    if self.b > self.Ef(self.r) and len(self.l) == self.size:
        return self.l.index(self.b)
```

3.3.1 Simulation of Mudae

With the model complete, we now work on simulating Mudae. Our function will be very similar to the model, starting by calling `reset` on the model and initializing its own variables, `l` to hold kakera values, and r, the number of rolls remaining, starts at $r_{\max} = 30$. We define `offset` as $R\%B$ again, and initialize `size` identically to the model, fulfilling the assumption that the offset is removed first.

simulate header

```
def simulate(m: model.Model=model.Model()) -> float:
    """ Simulates a game. """
    m.reset()
    l, r, offset = [], prob.R, prob.R % prob.B
    size = offset if offset > 0 else prob.B # remove the offset first
```

We get a kakera value k from the kakera random variable X with `sample`. We add k to our list and call the model's `update` function, which returns an index i if it is claiming, `None` otherwise. If i is an index, we make sure it is valid and return the kakera value it corresponds to. Otherwise, we continue and decrement r since we just simulated a roll cycle. Once we run out of rolls, we return 0 since the model didn't claim.

body of simulate

```
    while r > 0:
        # new batch, reset seen and assume offset has been taken care of
        if len(l) == size:
            l, size = [], prob.B
        k = sample()
        l.append(k)
        i = m.update(k) # give kakera value to the model
        if i is not None:
            assert 0 <= i < len(l), "model did not give a valid index"
            return l[i]
        r -= 1
    return 0 # model didn't claim, kakera value of 0
```

We can now verify that our math is correct by repeatedly running `simulate` for a large number of times, perhaps 10^6. The expected value is then simply the average value, the sum of the values `simulate` returns divided by the number of trials.

empirical expected value via the law of large numbers

```
def E(X, iters: int=10**6) -> float:
    """ Expected value by repeatedly sampling a random variable. """
    return sum(X() for i in range(iters))/iters
```

This confirms that the model expected value is 306.064, which is significantly higher than the expected value of the kakera distribution, 61.702. It is lower than $E[X_{30}]$, which is 342.126. X_{30} corresponds to if one was able to do all 30 rolls and then pick the highest value. Our model, only able to do batches of 10, is surprisingly close, however.

3.4 `$rolls` Model

We arrive at the last consideration in our model, the command `$rolls`. `$rolls` will reset the number of rolls in the current batch, giving 10 rolls if the user is currently out of rolls. However, the usage of `$rolls` is limited to once per day. Given that we can claim every 3 hours, and therefore claim 8 times a day, it follows that the model must use `$rolls` with less than a $\frac{1}{8} = 12.5\%$ probability. We separate our discussion into two parts: the first on simply implementing this new command in our model and simulation, and the second on the optimal determination of when to use the command.

3.4.1 Implementation

Recall that the model emits an index if it wants to claim, `None` otherwise. We define the constant `ROLLS` $= -1$ which the model returns if it wants to use `$rolls`, so there is no ambiguity between resetting and claiming (indexes are nonnegative integers). We also define `ROLLS_CYCLE` to be 8, the number of claim cycles until the model receives a new usage of `$rolls`, and `ROLLS_F` to be the probability it uses `$rolls`, $\frac{1}{\texttt{ROLLS\_CYCLE}}$. Finally, we assume the model only uses `$rolls` once per claim cycle. The actual game limits usage to once per *interval*, which is equivalent to a batch in our formulation. [3] It very rarely makes sense to use `$rolls` *before* we are actually out of rolls; we would have to anticipate that at least 2 straight batches are bad (10–0, the first usage of `$rolls`). [4]

The first thing we define is a helper function adjusting the state of the model after a use of `$rolls`. Since `$rolls` *resets* the current batch (not necessarily adding 10 rolls), we first create two helpful methods for modulo arithmetic. Any number x under $\mod n$ can be written in the form $qn + r$, or as a multiple of n and a remainder. The remainder r is the *modulo* of x, $x \mod n$ or $x\%n$.

When we reset a batch of size x, we are essentially finding the smallest multiple of the batch size B that is greater than x. To find the largest multiple of B that is *smaller* than x, since x can be written as $qB + r$, the lower bound is just qB. The upper bound is thus $qB + B$ or $(qB + r) + (B - r) = x + (-x \mod B)$. Since $B - r$ is between 1 and $B - 1$ if r is nonzero, $B - r = B - r \mod B = -r \mod B = -x \mod B$. If $r = 0$, then x is a multiple of B and is its own lower and upper bound, $-x \mod B = 0$.

```
def lower(x: int, b: int=B) -> int:
    """ Returns the  largest value n such that n <= r and n % B == 0. """
    return x - (x % b)

def upper(x: int, b: int=B) -> int:
    """ Returns the smallest value n such that n >= r and n % B == 0. """
    return x + (-x % b)
```

If we use `$rolls`, the amount of rolls we gain is the upper bound of $r + 1$, since if r is a multiple of B we want it to reset to the higher multiple of B, not at its own value. We set $\Delta = \texttt{upper}(r) - r$, and update r to $r + \Delta$ to put it at its new value. We update `size` as well to `size` $+ \Delta$, because our batch now contains the characters currently in the batch along with the new characters we roll from the rolls we got from `$rolls` (assuming we do all our rolls within the minute long grace period). Finally, since we only use one `$rolls` per claim, we set `rolls_use` to `False`, barring future usage of `$rolls`.

[3] If this limitation did not exist, check subsection 9.3 in the appendix for an analysis.

[4] It can be shown that the best return from a second use of `$rolls` is when the rolls from 10–0 are as bad as possible; the return is then $\mathrm{E}[Z_{20}] - \mathrm{E}[Z_{10}] \approx 86$. Comparing to the worst return for one use of `$rolls`, it can be shown that that occurs at the highest value which triggers a usage of `$rolls`, or 72. In that case the difference is $\mathrm{E}[\max(Z_{10}, 72)] - 72 \approx 122$. Because the best possible return for a second use is less the worst possible return for one use, it is never worth it to use `$rolls` twice.

helper function after using $rolls

```
def rolls(self) -> None:
    """ Update the model's parameters if it uses $rolls. """
    delta = prob.upper(self.r + 1, self.B) - self.r
    self.r += delta
    self.size += delta
    self.rolls_use = False
```

With the model adjusted, we now focus our attention on the simulation. We first need to give rolls to the model, which we will store in the `rolls_left` instance variable. We could give the model a roll with $\frac{1}{8}$ probability, but it is more consistent and accurate to the game to simply give the `simulate` method an integer tracking the game number, and give a usage if the index is a multiple of 8.

new simulate header

```
def simulate(m: model.Model=model.Model(), index: int=0) -> float:
    """ Simulates a game. """
    m.reset()
    # give the model a roll every ROLLS_CYCLE claim iterations
    if model.ROLLS_AVAILABLE and index % model.ROLLS_CYCLE == 0:
        m.rolls_left += 1
    ...
```

The only other change to make is when handling output from the model. If the output is not `None`, we know that the model is performing an action and handle the action appropriately. Claiming is handled the same, and we handle `$rolls` according to the `rolls` method defined above for the model.

new simulate body

```
    ...
    i = m.update(k) # give kakera value to the model
    if i is not None:
        if i == model.ROLLS:
            assert model.ROLLS_AVAILABLE, "$rolls is not allowed"
            assert m.rolls_left > 0, "model doesn't have available rolls"
            # $rolls resets the batch instead of adding the batch size
            delta = prob.upper(r, prob.B) - r + 1
            r += delta
            size += delta
            m.rolls_left -= 1
        else:
            assert 0 <= i < len(l), "model did not give a valid index"
            return l[i]
    ...
```

3.4.2 Optimization

With the framework of `$rolls` complete, we now discuss the determination of when to use the command. Very simply, optimization problems are always determined by two things: what we want to optimize and what the constraints are. The constraint is that we can only use `$rolls` every one in 8 claim cycles. We want to optimize kakera, so we should use our limited number of `$rolls` on the situations that maximize the gained value. This is equivalent to being in the *worst* situations—the worse the situation, the more advantage additional rolls gives us (put another way—the better the situation, the less probability `$rolls` can improve on the situation).

Going from the abstract to the more specific, the measure of the quality of a situation is simply its kakera value. If we're about to claim a low kakera value character, that is a situation to avoid. So we should use `$rolls` for any kakera value less or equal to a certain cutoff, k^*. How should k^* be determined? Well, suppose we have the CMF of the model, $F_{\text{model}}(k)$, which gives the probability we claim a value less than or equal to k. Because we use `$rolls` whenever we're about to claim a value less than or equal to k^*, the probability we use `$rolls` is $F_{\text{model}}(k^*)$, which must be $\leq \frac{1}{8}$. Therefore, we need to compute F_{model} in order to compute k^*.

In order to compute F_{model}, we first need to compute the probability of emitting a value at a particular number of rolls left r, $p(\text{emit at } r)$. To compute that, we first need to compute its CMF, $p(\text{emit at} \leq r)$. I know, very straightforward. Suppose we know the probability of getting to $r+1$ rolls left is $F(r+1)$. Then the probability of getting to r rolls left is if we don't emit at roll $r+1$. We don't emit if the value rolled is less than the expected value cutoff, or if $Z_B \leq \mathrm{E}[X_r]$. By definition, $p(Z_B \leq \mathrm{E}[X_r])$ is the CMF of Z_B, a value we'll call p. So $F(r) = pF(r+1)$, and the base case is that $F(30) = 1$ (since the largest number of rolls we have is 30). Note that if r is not a multiple of the batch size, $p = 1$ since we only claim at the end of a batch.

In order to implement this, we first define a useful function which gives the current Z_n (since if the number of rolls isn't a multiple of the batch size, we'll have partial batches). We use Z_B by default, but if we're in the offset (if $r \geq R - \text{offset}$) then we use Z_{offset}.

current random variable

```
def __Z(self, r: int) -> list:
    """ Returns the correct cmf of Z_n, accounting for offset. """
    return prob.Fzs[self.offset if r >= self.R - self.offset else self.B]
```

The CMF then follows from the discussion above and the PMF is simply the adjacent difference of the CMF.

cmf and pmf of emitting a value at r rolls left

```
def F_r(self, r: int) -> float:
    """ Probability of getting to roll r. """
    # since we never claim in the middle of the batch, use Z instead of X
    p = prob.cmf(self.__Z(r), self.Ef(r)) if r % self.B == 0 else 1
    return 1 if r == self.R else p*self.F_r(r + 1)
```

```
def p_r(self, r: int) -> float:
    """ Probability of emitting a value at roll r, pmf of F_r. """
    return self.F_r(r) - self.F_r(r - 1)
```

We now compute the probability of emitting a kakera value k. We can do casework on each possible number of rolls left r thanks to conditional probability, a quick description of which is given in the appendix, subsection 9.2.

$$p(\text{emit value } k) = \sum_{r=0}^{R} p(k|r) \cdot p(r)$$

$p(r)$ is the function defined above and $p(k|r)$ is simply the probability we emit k given we have r rolls left. We emit if we roll a value *higher* than the cutoff, so the denominator will be the survival function while the probability we emit k is the PMF, assuming k is greater than the cutoff. If it isn't, k has a probability of 0.

$$p(k|r) = \frac{p_{Z_B}(k)}{1 - F_{Z_B}(\mathrm{E}[r])}$$

Rewriting $p(\text{emit value } k)$ using $p(k|r)$,

$$p(k) = p_{Z_B}(k) \sum_{r=0}^{R} \frac{p(r)}{1 - F_{Z_B}(\mathrm{E}[r])}$$

Note that the sum is a function of r and not k, so it can be cached for different k. But we can't just store the right sum as a scalar because if k is less than the cutoff at a particular level, its probability is zeroed out. We therefore store the prefix sum and binary search on the cutoffs to find the level r' such that $k \geq \mathrm{E}[X_{r'}]$. We also need to account for a possible offset by switching p_{Z_B}, so we in fact maintain two prefix sums, one for r below the offset and the other for r above the offset.

calculating the pmf of emitting a value k

```
def __cache_p_k(self) -> None:
    """ Generate cache for p_k """
    self.poss = [1 - prob.cmf(self.__Z(r), self.Ef(r))
                 for r in range(self.R)]
    i, f = self.R - self.offset, lambda r: self.p_r(r + 1)/self.poss[r]
    self.cond = prob.prefix_sum(list(map(f, range(i))) + [0]*self.offset)
    self.coff = prob.prefix_sum([0]*i + list(map(f, range(i, self.R))))

@lru_cache(maxsize=None)
def __p_k(self, k: int) -> float:
    """ Probability of emitting the kakera value k. """
    i = min(bisect.bisect(self.E, k), self.R)
    return self.fz(k)*self.cond[i] + prob.fz(k, self.offset)*self.coff[i]
```

With $p(k)$ computed, we now have F_{model} which if you will recall, we use to find the largest value k^* such that we will use `$rolls` if we're about to claim a value $\leq k^*$. We simply binary search on the CMF to find the value with CMF greater than $\frac{1}{8}$, and subtract 2 (one for the CMF, and one to make it less than).

the rolls model

```
def __init__(self, R: int=prob.R, B: int=prob.B,
  ...
  if ROLLS_AVAILABLE:
      # generate cmf of p_k
      self.Fk = prob.prefix_sum(list(map(self.__p_k, prob.X)))
      # the largest value k* such that Fk(k*) triggers the roll cutoff
      self.kp = prob.X[bisect.bisect(self.Fk, ROLLS_F) - 2]
      assert prob.cmf(self.Fk, self.kp) <= ROLLS_F, "valid cutoff"
      self.rolls_left = 0

def update(self, k: int) -> int:
    ...
    # no need to stop early, wait until the batch is complete
    if self.b > self.Ef(self.r) and len(self.l) == self.size:
        # use $rolls if emitting this bad of a value has probability 1/8
        if self.rolls_use and self.rolls_left > 0 and self.b <= self.kp:
            self.rolls()
            return ROLLS
        return self.l.index(self.b)
```

Finally, we can recompute the PMF for the rolls model, since using rolls will change the behavior of our model. To simplify the calculations, we assume that $k^* \leq \mathrm{E}[X_B]$, an assumption which can be checked empirically. Intuitively, the more `$rolls` we get, the larger k^* will be since we can use it more, and the less uses we get the smaller k^* will be. $\frac{1}{8}$ is sufficiently small to force $k^* \leq \mathrm{E}[X_B]$, which means that we only use `$rolls` with no rolls left (E is monotonically increasing with the number of rolls left, therefore for a larger number of rolls left, if we claimed, we must have claimed for a value $> k^*$). Thus, the PMF is only different for the last layer. We therefore look at $p_{\text{last}}(k)$, the probability of emitting the kakera value k at the last layer. The old PMF was simply Z_B. For the new PMF, we sample Z_B once. If the value is $\leq k^*$, we sample Z_B again. We now analyze two cases, $k > k^*$ or $k \leq k^*$ which are independent and cover all cases.

Let's say we're looking at a value $k > k^*$. Then we could have rolled it in the initial sample and stopped, or the first sample was $\leq k^*$, so we did a second sample and rolled k. Since the cases are independent, we add their probabilities:

$$p_{Z_B}(k) + F_{Z_B}(k^*)p_{Z_B}(k) = (1 + F_{Z_B}(k^*))p_{Z_B}(k)$$

In the case that $k \leq k^*$, we must have that k is the largest value in two separate samples of Z_B, or simply the largest sample of Z_{2B}. Thus, the probability is $p_{Z_{2B}}(k)$.

last layer pmf of the rolls model

```
def p_last(self, k: int) -> float:
    """ Probability of emtting the kakera value k on the last layer. """
    return (1 + prob.cmf(self.Fz, self.kp))*self.fz(k) if k > self.kp \
            else prob.fz(k, 2*self.B)
```

We now compute the overall PMF by simply swapping out the last layer PMF.

$$p_{\text{new}}(k) = p(k) + p(\text{emit at layer 1})(p_{\text{last}}(k) - p_{Z_B}(k))$$

pmf of the rolls model

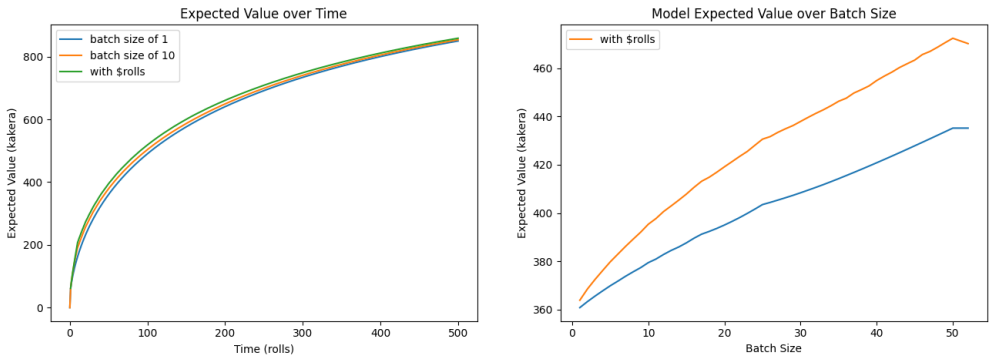
```
def __rolls_p_k(self, k: int) -> float:
    """ Probability of emitting the kakera value k. """
    return self.__p_k(k) + self.p_r(1)*(self.p_last(k) - self.fz(k))
```

With the mathematical analysis of the rolls model complete, we switch to a quantitative and qualitative evaluation. Starting with the expected value:

Name	Expected value (kakera)	Standard deviation (kakera)
Kakera	61.702	79.727
Z_{10}	191.873	196.072
Z_{30}	342.126	259.412
Model	306.064	255.263
Rolls Model	322.036	252.524

Table 1: Table of random variables.

Unsurprisingly, the expected value of the rolls model is slightly higher, since it's sometimes able to avoid low kakera values. Looking at the graphs confirms this trend.



(a) Expected value over rolls left.

(b) Expected value over batch size.

Figure 8: Expected value graphs.

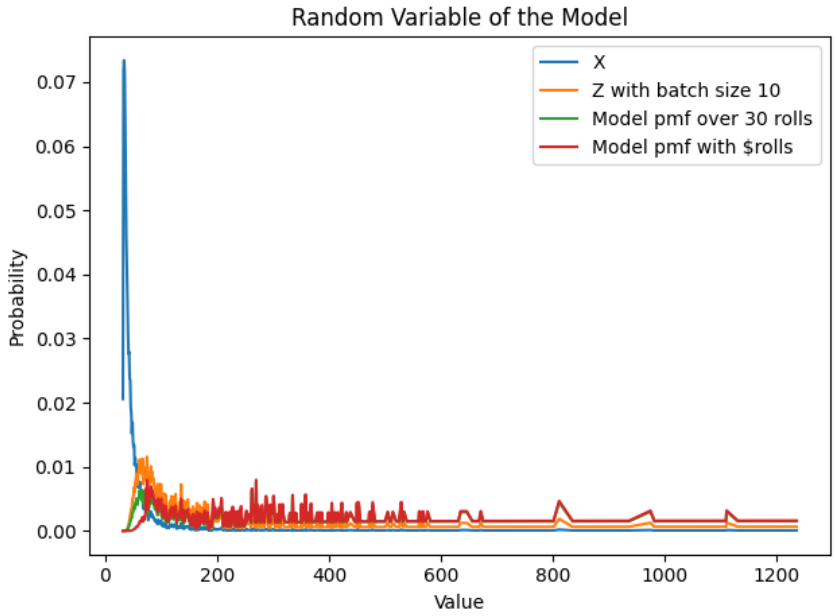


Figure 9: Model PMFs

Since we have the PMF, we can now graph the model PMFs: Towards the left side, the rolls model has a much lower slope before rising parabolically upwards because it is able to reduce the probability of low kakera values. One interesting observation is that the local maxima in the original distribution are magnified: see the spikes at 700, 800, 1000, and 1100. With the PMFs we can also analytically calculate the variance:

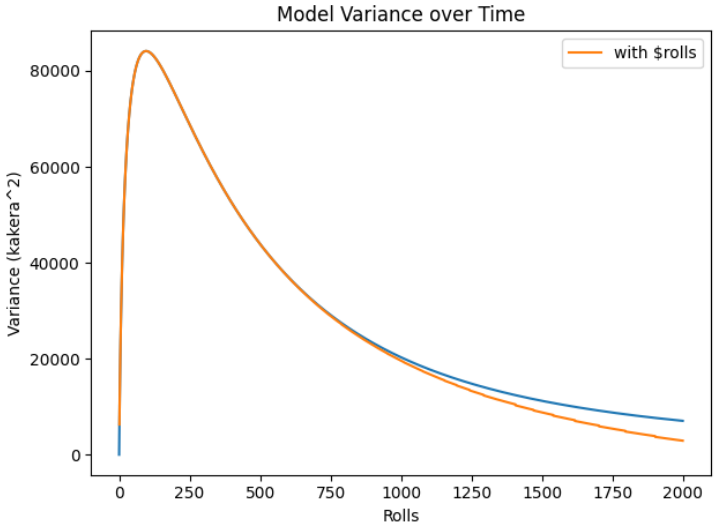


Figure 10: Variance over rolls left.

This has a surprisingly beautiful shape, coming up to a local maxima before smoothly asymptotically approaching zero. The variance decreases because more rolls essentially means more samples, but I'm not sure why the variance increases towards the beginning. Whatever the case, it sort of resembles a blackbody radiation curve so there may be some connection to statistical thermodynamics.

Finally, we can graph variance over batch size:

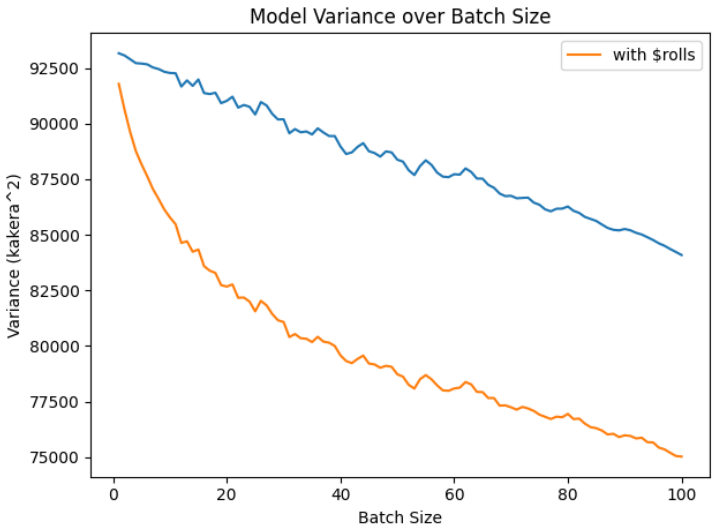


Figure 11: Variance over batch size.

Without rolls the variance is suspiciously linear, which is quite surprisingly for such a nonlinear model. With rolls the variance starts to curve, exposing the underlying nonlinearity. Why increasing batch size has a seemingly linear effect on both expected value and variance when the PMF changes with respect to the power of the batch size is hard to determine. With the rolls model complete, we now analyze one last mechanic.

3.5 Roll Pricing

Recall we only claim at the end of a batch, never in the middle of a batch. Doesn't this contradict our maxim "Claim when the current value is greater than the expected value of r rolls left"? Well, what principle justified our maxim in the first place? We have two options, claim or continue rolling. If we claim, the value is the value of the character. If we continue rolling, the value is the expected value of r rolls left. We want to maximize value, so we pick the option which gives the larger value. But hold on a second — we have more specific information than just r rolls left! In particular, if we're in the middle of the batch then we have some largest value k. The value of our current batch is then lower bounded by k. Thus, the value of continuing the batch is a capped expected value, $\mathrm{E}[\max(Z_n, k)]$ where Z_n is the appropriate batch random variable and k is the current largest value in the batch, since if we roll values lower than k we can always claim k.

expected value of continuing

```
def Eloss(self, r: int, k: int) -> float:
    """ Expected value if the current batch is continued. """
    n = r % self.B
    return prob.capped(prob.Z, prob.Fzs[n], prob.evzs[n], k)
```

The real value of our current state, or our "status quo" is then the maximum between E_{loss} and $\mathrm{E}[X_r]$, since we either claim in the current batch, or do our r remaining rolls.

```
    def status_quo(self, r: int, k: int) -> float:
        """ Represents the current value. """
        return max(self.Eloss(r, k), self.Ef(r))
```

We could therefore rewrite the rule of "claim when the value is greater than the expected value and only when the batch is over" as "claim when the value is greater or equal to the status quo". If we're in the middle of a batch, E_{loss} is greater than k, so we're never going to claim[5]. If we're at the end of a batch, then $r\%B$ is 0, $\mathrm{E}[\max(Z_0, k)] = k$ so E_{loss} reduces to k, so checking whether k is greater than the status quo reduces to checking whether k is greater than $\mathrm{E}[X_r]$. Why did we derive this alternative representation of our current value when our more computationally efficient stopping rule works just fine?

We can now define *opportunity cost*, the basis of value I will use to justify the formulation of judging the value of buying or selling rolls. Since other users are also on our Discord server playing Mudae, we can choose to buy or sell rolls. Suppose we used all our rolls and didn't get a good value. Then we can pay someone else to roll for us, and claim from their rolls. Suppose we claim a high value character, and have rolls left. Then we can sell our rolls to someone else, and let them claim from the rolls that are now useless to us. How should buying and selling be priced?

We first analyze buying. If we buy b rolls, then our new expected value is $\mathrm{E}[X_{r+b}]$ compared to $\mathrm{E}[X_r]$. If we were at a new batch then the price at which we should buy these b rolls would simply be $\mathrm{E}[X_{r+b}] - \mathrm{E}[X_r]$ since we gain that much expected value from using the b rolls. However, if we're in the middle of a batch we must use the derived status quo calculation. If our status quo is more valuable than $\mathrm{E}[X_{r+b}]$ then the extra rolls don't hurt—we should price them at 0.

buy price

```
def __buy(self, r: int, k: int, b: int) -> float:
    """ Finds the price which we are indifferent to buying b rolls. """
    return max(self.Ef(r + b) - self.status_quo(r, k), 0)
```

Selling is very similar. If we sell, then we must claim immediately, so our current value is k, the largest value in the current batch. If we didn't sell, then the value would

[5] If k is the largest possible value, then the capped expected value will not add to its value, so we would claim k immediately instead of finishing the batch — it's not possible to improve.

be the status quo. This elegantly accounts for the case where the value of the status quo is k, in which case we are indifferent to selling and price our sales at 0.

sell price

```
def __sell(self, r: int, k: int) -> float:
    """ Finds the price which we are indifferent to selling r rolls. """
    return self.status_quo(r, k) - k
```

In order to visualize the pricing for buying and selling, we assume we're at a start of a new batch. In that case the value of buying is simply $\mathrm{E}[X_{r+b}] - \mathrm{E}[X_r]$. If we're buying one roll, then this is simply the adjacent differences of the expected value graph.

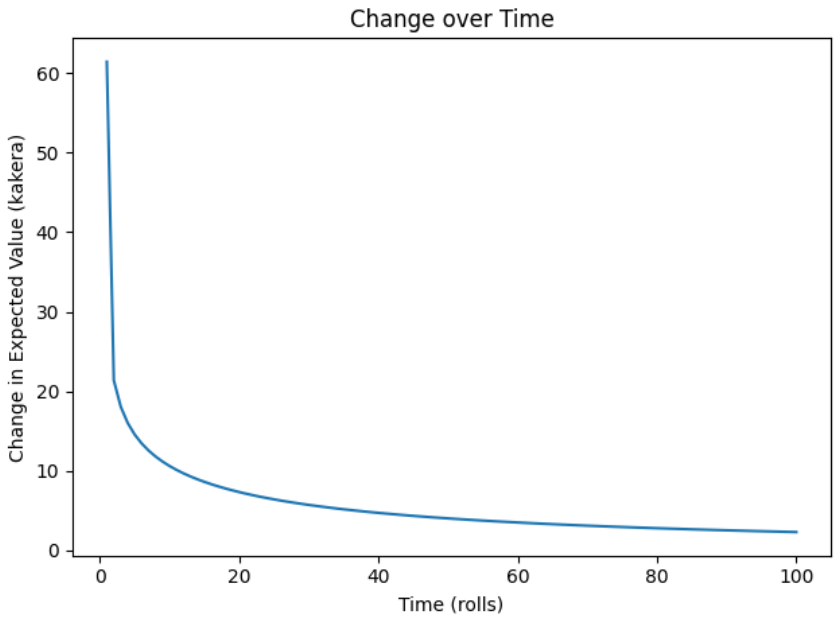


Figure 12: Discrete derivative of the expected value graph.

The value of buying a single roll sharply drops as the number of rolls left increases.

4 Convolutions and the Central Limit Theorem

This section is a mathematical tangent, feel free to skip to the conclusion.

Suppose we know that we will do 5 claim cycles. By the linearity of expected value, the expected value will be 5 times the expected value of one claim cycle, but can we figure out the PMF of the new random variable?

4.1 Derivation of the Sum of i.i.d Random Variables

We have two independently and identically distributed random variables X and Y and wish to find the PMF. Surprisingly enough, we can directly compute the PMF without any CMF or continuous trickery. Suppose we want the probability of the value v occurring. Then we can split into cases based off X's values. If X takes on some value i then Y must take on the value $v - i$ for the value of X plus the value of Y to be v. Since X and Y are independent, the probability of them taking their respective values is simply the product of the probabilities, and we sum over all possible values of i.

$$p_{X+Y}(v) = \sum_{i=-\infty}^{\infty} p_X(i)p_Y(v-i)$$

We call this function the ***convolution*** of the PMFs p_X and p_Y, denoted $p_X * p_Y$.

Theorem 4.1. *The discrete convolution is equivalent to polynomial multiplication.*

Proof. We assume the discrete random variables takes on nonnegative integer values.

We encode a random variable X into a polynomial a using the following: First, we represent polynomials as a coefficient list. For each value X can take, we index a at that value and put the probability it occurs at that index. Put another way, if x is a value of X occurring with probability p, then we make $a[x] = p$.

If a and b are polynomial representations of X and Y, then the convolution is:

$$(a * b)[n] = \sum_{i=0}^{n} a[i]b[n-i]$$

This is equivalent to the polynomial multiplication of a and b. □

Along with computing the probability function of the sum of two random variables, the convolution has many, many uses:

- Signal processing, audio processing (spectrograms)
- Polynomial multiplication, integer multiplication
- String matching
- Differential equations: the Laplace transform of the convolution is the product of the Laplace transforms, $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$
- Deepfake detection (really!)
- Kernels in image processing
- Gives "convolutional" neural networks their name

We do not discuss how to compute the convolution quickly; I have a lecture. It can be done in $O(n \log n)$ with the Fast Fourier Transform (FFT), compared to the naive polynomial multiplication of $O(n^2)$. Instead, we simply apply the convolution to compute the PMF of the sum of i.i.d. random variables.

4.2 The Central Limit Theorem

The central limit theorem is one of the most surprising theorems in probability; it states that the distribution of the average value of multiple samples of a random variable approaches a normal distribution as the sample size approaches infinity, for any random variable.

As proved previously, the sum of i.i.d random variables is the product of their polynomial forms. Since each random variable has the same polynomial form, it is the power of the polynomial. One could use a generalization of the binomial formula, the multinomial coefficients to compute this power, but it is simplest to use the fast exponentiation trick, for $\log k$ multiplications, using the convolution for each polynomial multiplication. This yields a $O(kn \log kn)$ time algorithm where k is the number of samples and n is the length of the polynomial, compared to $O(k^2 n \log kn)$ without fast exponentiation and $O(k^2 n^2)$ without either trick. The lower bound is $O(kn)$.

We can directly apply this onto the PMF of the kakera distribution:

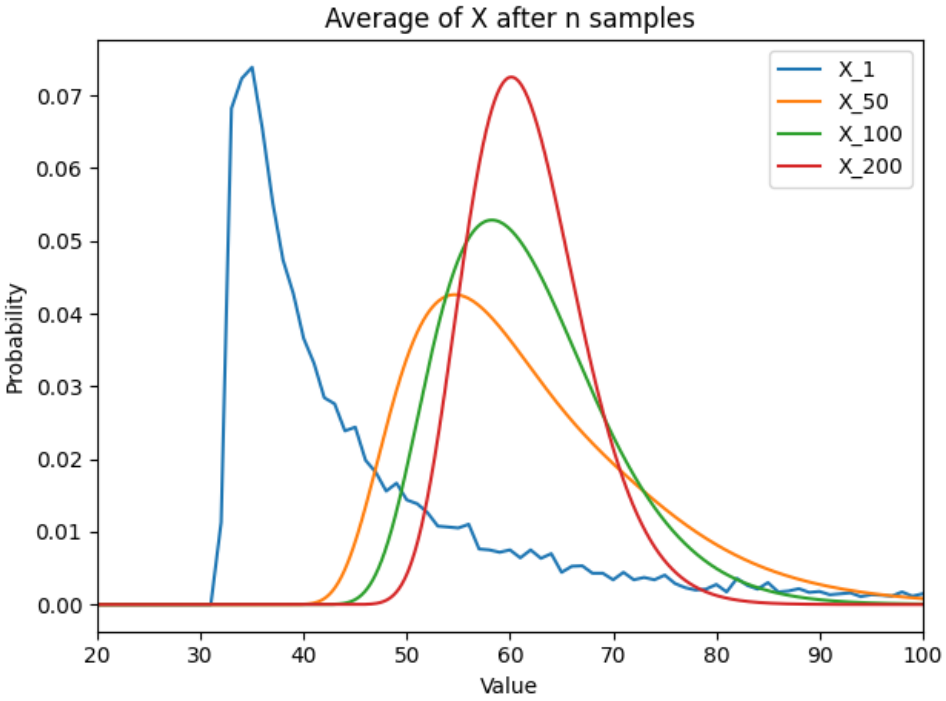


Figure 13: A demonstration of the central limit theorem.

As the number of samples increases, the graph becomes more normal. We see the long right tail disappear as the PMF becomes more symmetric, the rightness being absorbed in the peak shifting to the right. The variance also lowers, as the graph becomes tighter and the peak grows higher. The peak will eventually reach the expected value of the distribution, 61.702 while the graph becomes narrower and narrower.

5 Is the Model Realistic? "Optimal"?

Throughout this paper, we have been focused on maximizing expected value. I claim that the model is optimal with respect to its expected value. But is expected value really the thing we want to maximize?

Suppose I offer you a 50% chance to get a million dollars versus a 80% chance for $500,000. Which should you pick? Despite the higher expected value for the million dollars, it is not obvious. One can argue the difficulty is psychological in nature or related to the diminishing utility of money rather than the pure mathematically "rational" action to take. But it is hard to separate.

Because of the strong law of large numbers, taking the average of many samples of a random variable will converge to the expected value, as the number of samples approaches infinity. Expected value is then essentially the value over an infinite *timescale* or *horizon*. Generally speaking, the lower the variance, the faster the average value will converge to the expected value. But even if the variance is infinite it will still converge. Is our timescale really infinite in the real world? Are we going to play this Discord game forever? The timescale really does matter, and with it what we define as "optimal".

For example, take the St. Petersburg paradox. The expected value of the game is infinite, but if we pay $50 we have only about a 3% chance to get our money back. If we play the game *infinite* times, yes, the expected value will converge to infinity. But if we play only once or twice, should we really sell everything we own and take out loans for the opportunity to play this game? The St. Petersburg paradox poses a fundamental threat to the notion that maximizing expected value is the "rational" thing to do.

One way to fix this it to notice that if we don't know the rules of the game and need to figure them out as we play, then we don't know what hasn't happened — events with low probability aren't likely to happen, so we won't even know they exist. If we only play once or twice, we might guess that the expected value is around 4. So one rule would be "disregard low probability events". This elegantly works if we play the game multiple times, since the more times we play the game the more likely low probability events are to occur, which might move us over the arbitrary "low probability" cutoff. But there is a clear weakness: suppose we disregard any events with a probability $\alpha = 0.05$ or lower. I now offer you the following game: I give you a million dollars, except I also hand you a slip of paper with a number between 1 and 20 written on it. The chance that any particular situation occurs is $\frac{1}{20} \leq \alpha$. So you conclude the value of the game is 0 since no situations occur! If I wrote a real number on the slip of paper, then no α is low enough. So disregarding low probability events is clearly not a general principle.

Well, instead of comparing values what if we duel the two strategies? We compare the strategy of "don't play the game" versus the strategy of "play the game at some cost" and simply pick the one that is more likely to win. But this also has a clear failure point — it doesn't account for the *magnitude* of a win. I offer you a 100% chance for 1 dollar versus a 49% chance for 1 million dollars. Let's say I take the dollar while my friend aims for the million dollars. The chance I beat my friend is going to be 51% while the chance my friend beats me is 49%. I am clearly more likely to win, so I should take the dollar!

Of course, while the million dollars is slightly less likely, it gives a much larger reward when it does happen. So comparing strategies naively is also not a general principle.

Let us conclude by going back to Mudae. We have spent 34 pages describing exactly how to maximize expected value, only to just now question the entire basis of our modeling. It is to temper expectations. Many people have told me that the model is too optimistic and is not aligned with the reality of the game. I sometimes feel the same way myself. Ignoring psychological and irrational explanations, one reason is that expected value is not a perfect metric. But it is good enough.

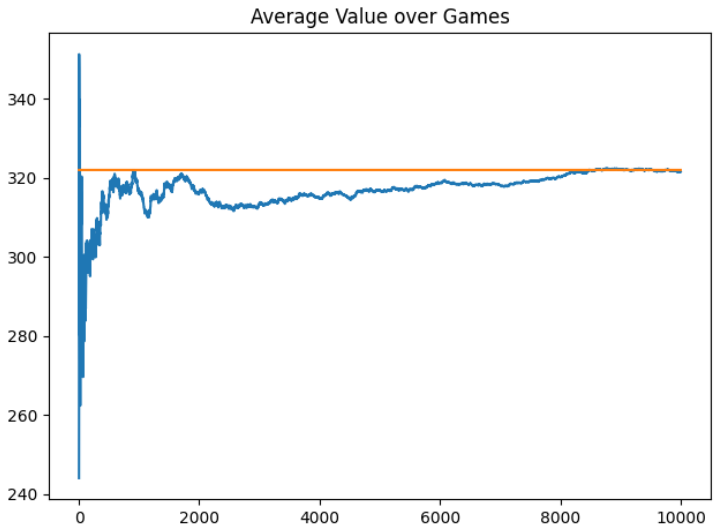


Figure 14: Average value over number of games.

Yes, it sometimes takes many games (in this case, about 10,000) to converge to the expected value. But the strong law of large numbers guarantees it, and because the variance is not too large it is not too unreasonable. I personally have about 3,000 rolls over the period of about a month, and anonymous others have much more than me.

Suppose we knew we would play exactly 10,000 games and then stop. Could we explicitly compute the PMF of the model after 10,000 games with convolutions and then pick the model that had the highest probability of winning at the end? Yes, but would we want to? Would such a thing be tractable?

I conclude on a bit of life advice. It is popular to have phrases like "carpe diem" (seize the day) or "you only live once" (YOLO). A thought experiment is to live each day like it's your last. But as we've seen, the concept of *timescale* is very important. If your timescale is very short, a rational person would never invest since the returns would not pay off in time. Expected value is investment over an infinite timescale, which might be extreme in the other direction. Find a happy medium.

6 `$disablelist` Optimization

With the optimal stopping rule derived, we now focus on a nearly completely orthogonal direction, optimizing *disablelists*. We first need to define a few terms. A *character* is the basic unit of Mudae, the objects that are rolled. A *series* is a set of characters. It can be verified that the series are independent, that is, there is no pairwise intersection between any two series. Finally, a *bundle* is a set of series. Bundles can have intersections. We are allowed to disable bundles or series, that is, to prevent the characters they contain from being rolled, but with a few restrictions. We can only disable up to 10 bundles or series, and the sum of the number of characters in each has to be less than 20,000, which is confusingly called the "overlap limit" — it does not actually account for overlap; it simply adds up the number of characters in each bundle or series regardless of whether a character appears multiple times. For simplicity, we only consider disabling bundles (we can disable series by creating pseudo-bundles containing a single series). Finally, the list of bundles we have disabled is called a *disablelist*. We therefore consider how to pick the right disablelists for two objectives—acquiring characters and maximizing kakera. [6]

6.1 Cardinality Optimization

We start by trying to acquire characters because it is much simpler. Suppose one desires a particular character for emotional reasons irrespective of their kakera value. In order to increase the likelihood they roll this character, they can disable as many characters as possible to decrease the denominator of possible characters, increasing the probability.

We have to be careful when we say "character". There are in fact multiple types of characters, and using the `$wa` command specifies a particular type of character to roll from. However, the overlap limit sums up the size of the bundle, measured by any type of character the bundle contains. The problem is therefore to minimize the number of wa characters (and therefore maximize the number disabled) subject to the constraints of disabling 10 bundles and respecting the overlap limit.

Ignoring the overlap limit for now, this problem is identical to maximum set cover. The max set cover problem is given a list of sets, pick K sets to maximize the number of elements covered, or to maximize the cardinality (length) of the union of the sets.

If there was no overlap between the sets, then a simple greedy algorithm which picked the largest set at every iteration would be correct. However, there is overlap, so the greedy is not correct. If we wanted a optimal solution, we could brute force all the possible ways to pick K sets from N, $\binom{N}{K}$, which is prohibitively large for large N or K. Can we therefore use greedy to get a pretty good, but not optimal, solution?

[6] We focus on optimizing personal disablelists, but the server can disable things as well. In particular, my server had `$togglewestern` and `$toggledisturbing` on. There are also two other toggles, but they were not disabled. I mention this because the optimal solution will have to take into account what the sever has disabled—if the server disabled a character, you disabling it as well doesn't do anything.

6.1.1 Greedy and Approximate Algorithms

We can be a bit more clever with the greedy by picking the set which adds the most new elements, taking into account overlap, instead of just the largest set out of the sets we haven't picked. This still isn't optimal, but thanks to a clever proof, this yields a $1 - \frac{1}{e} \approx 0.63$ approximation algorithm. That means that in the worst case, the size of the union of the sets greedy picks will be 63% of the optimal solution.

That seems decent, but we still need to take into account the overlap limit. Unfortunately, this is were the beautiful math goes to waste because there's no easy way to take it into account. I tried a few heuristics, but my best was prioritizing the number of wa characters disabled towards the start, and the size of the bundle as the algorithm gets closer to the overlap limit. If W is the number of wa characters, N the number of characters, and D the current number of disabled characters, then I picked the bundle with the best heuristic value:

$$W + \frac{3}{4}e^{D/20000-1}(-N)$$

That yielded 12,528 characters disabled compared to the 12,658 disabled optimally, which is much better than the 0.63 worst case approximation—it's nearly 0.99.

Because the problem is NP-hard (intuitively, pretty hard), greedy is pretty much the best we can do. Or is it? How did I calculate the optimal solution for 1,086 bundles, 7,367 series, and over 15,580 characters?

6.1.2 Mixed-Integer Linear Programming (MILP)

A classic mistake is thinking that NP-hard problems are unsolvable. NP-hard means hard in the *worst case*. In practice, clever heuristics and algorithms make the *average case* quite tractable, especially if one uses tried-and-tested off-the-shelf programs.

Linear programming (LP) is a class of optimization problems defined by a linear objective and linear constraints. *Integer linear programming* (ILP) means the variables are restricted to integers. Finally, *mixed-integer linear programming* (MILP) means the variables can be integer or continuous. LP can be solved in polynomial time, but ILP and MILP are NP-complete. However, there are out of the box solvers which can solve MILP problems with thousands of variables and constraints; I use the python-mip package which leverages the COIN-OR Cbc solver. Many problems can be stated as linear programs, so it useful to formulate a problem as linear programming, then use an external program to solve the resulting linear program. This is easier and likely faster than writing a ad-hoc algorithm for the particular NP-hard problem, and can easily be extended (in this case, to account for the overlap limit).

We can formulate our problem as integer linear programming. Suppose x_i is a binary variable (it is an integer variable which takes on the value 0 or 1) that represents whether the ith bundle is picked, s_i is the size of the ith bundle (what contributes to the overlap limit), y_i is whether the ith series is disabled or not, and finally w_i is the number of wa characters in the ith series. Then the program is:

$$\text{maximize} \sum w_i y_i \qquad \text{(number of wa characters)} \qquad \text{(cardiality program)}$$

subject to:

$$\sum x_i \leq K \qquad \text{(can only pick } K = 10 \text{ bundles)}$$

$$\sum s_i x_i \leq 20000 \qquad \text{(overlap limit)}$$

$$\text{for each } y_i\text{:} \sum_{j|y_i \in x_j} x_j \geq y_i \qquad \text{(if } y_i \text{ is disabled, at least one bundle disables it)}$$

$$x_i, y_j \in \{0, 1\} \qquad (x_i \text{ and } y_j \text{ are binary variables})$$

It is quite simple to turn this linear program into code. We start by defining the model, then make binary variables for each bundle and series.

model and variables

```
m = Model(sense=MAXIMIZE, solver_name=CBC)
# whether the ith bundle/series is disabled
x = [m.add_var(name=f"x{i}", var_type=BINARY) for i in range(N + A)]
# whether the ith series is included or not
y = [m.add_var(name=f"y{i}", var_type=BINARY) for i in range(M)]
```

We now add the objective, to maximize the number of wa characters:

objective

```
m.objective = xsum(w[i]*y[i] for i in range(M))
```

Finally, we add each constraint. We can only pick at most K bundles:

```
m += xsum(x) <= NUM_DISABLE, "number_disable"
```

We have the overlap limit to account for:

```
m += xsum(s[i]*x[i] for i in range(len(x))) <= OVERLAP, "overlap_limit"
```

The last constraint is the one preventing series from disabling themselves:

```
for i in range(M):
    bundles = [x[j] for j in range(N) if series[i] in bundle[j]]
    # the psuedo-bundle containing just the series
    bundles.append(x[i + N])
    # if yi is included, at least one bundle needs to have it
    m += xsum(bundles) >= y[i], f"inclusion{i}"
```

There are couple of things to note. When we disable a bundle, we never actually explicitly disable the series it contains. The program will naturally disable them to maximize the objective. Since the running time increases with respect to the number of constraints faster than the number of variables, avoiding extra constraints is good.[7] We can disable series and avoid going to the character level since series are pairwise independent, so the sum of the characters disabled will be simply the sum of the characters in each series. Inherent to the structure of the program itself, we elegantly avoid the need to explicitly union the bundles by creating the binary variable for each series. Thus if two bundles contain the same series, we do not over-count since that series's indicator variable will be 1, regardless of how many bundles disable it.

One trick we can use to speedup the program is to enforce that the number of bundles disabled is exactly equal to K instead of less than or equal to. It is possible to imagine situations where it is advantageous to not disable all the bundles we are allowed to, in cases where we are very close to the overlap limit so disabling an extra bundle would push us over, necessitating a change to our bundles. In practice, however, the program generates solutions with K bundles disabled, so forcing exactly K speeds up the search.

With our efforts, we are able to cut the number of characters by approximately π, increasing the probability by three-fold, reducing the number of rolls necessary to the get the characters we want. With many other people using my optimized disablelists, the effect is cumulative and we are rewarded from our hard work:

Figure 15: My collection.

[7] See page 18 of this chapter.

6.2 Mixed-Integer Linear Fractional Programming (MILFP)

With cardinality essentially solved, we switch our focus to maximizing kakera. We must first discuss theory to give context for our next program.

Recall we solved cardinality by writing the problem as a MILP. We now consider *fractional* programs, when our objective is a fraction with a linear numerator and a linear denominator. We will warm-up by first considering fractional constraints. Suppose we have a constraint of the form:

$$\frac{a_1x_1 + a_2x_2 + \dots}{b_1x_1 + b_2x_2 + \dots} \leq c$$

If we assume that the denominator has the same sign (we can assume positive, if the denominator is negative we can negate the denominator and negate the numerator), then we can multiply both sides by the denominator without changing the inequality.

$$\begin{aligned} a_1x_1 + a_2x_2 + \dots &\leq c(b_1x_1 + b_2x_2 + \dots) \\ (a_1 - cb_1)x_1 + (a_2 - cb_2)x_2 + \dots &\leq 0 \end{aligned}$$

The same trick works for $\geq$ or $=$, of course. If the constraint is equality, there does not need to be the requirement that the denominator keeps the same sign because the sign won't switch when multiplying across. If the denominator is 0, then the multiplication will be 0, forcing the numerator to be 0.

6.2.1 Charnes-Cooper

We now consider how to turn a fractional objective into linear programming. Suppose we want to maximize $\frac{a_0+a_1x_1+a_2x_2+\dots}{b_0+b_1x_1+b_2x_2+\dots}$ subject to constraints of the form $c_1x_1+c_2x_2+\dots \leq d$. We cannot do the same multiplication trick because there is only one side of the objective. We instead create a new variable $y_0 = \frac{1}{b_0+b_1x_1+b_2x_2+\dots}$ which represents the denominator. As usual, the denominator must stay the same sign and be nonzero. Using this new variable, we transform the objective:

$$\begin{aligned} \text{maximize } \frac{a_0 + a_1x_1 + a_2x_2 + \dots}{b_0 + b_1x_1 + b_2x_2 + \dots} \\ &= (a_0 + a_1x_1 + a_2x_2 + \dots)y_0 \\ &= a_0y_0 + a_1y_0x_1 + a_2y_0x_2 + \dots \end{aligned}$$

The problem is that we have variables multiplied by each other, which is not linear. However, we can be clever and substitute $y_i = y_0x_i$ in order to make the problem linear. Our objective has already been transformed, and for each constraint, we multiply on both sides by y_0 to force it into the terms of the newfound substitution.

$$\begin{aligned} c_1x_1 + c_2x_2 + \dots &\leq d \\ = c_1y_0x_1 + c_2y_0x_2 + \dots &\leq dy_0 \end{aligned}$$

Finally, we need to enforce the constraint that y_0 is the denominator. We can convert fractional constraints into linear constraints by multiplying across:

$$y_0 = \frac{1}{b_0 + b_1 x_1 + b_2 x_2 + \ldots}$$
$$b_0 y_0 + b_1 y_0 x_1 + b_2 x_2 y_0 + \ldots = 1$$

With the objective, constraints, and denominator prepared, we now make the substitution. The original program was:

$$\text{maximize } \frac{a_0 + a_1 x_1 + a_2 x_2 + \ldots}{b_0 + b_1 x_1 + b_2 x_2 + \ldots}$$

subject to:

$$c_1 x_1 + c_2 x_2 + \cdots \leq d$$

With the substitution $y_i = y_0 x_i$ this becomes

$$\text{maximize } a_0 y_0 + a_1 y_1 + a_2 y_2 + \ldots$$

subject to:

$$c_1 y_1 + c_2 y_2 + \cdots \leq d y_0$$
$$b_0 y_0 + b_1 y_1 + b_2 y_2 + \cdots = 1$$

We essentially get the same program! The only differences is that we change the objective to just the numerator, and multiply each constant by y_0. Lastly, we turn the denominator into a new constraint. In terms of the compact matrix-vector way to write a linear program, our original objective was essentially $\frac{\boldsymbol{c}_1^T \boldsymbol{x}}{\boldsymbol{c}_2^T \boldsymbol{x}}$ where the vector $\boldsymbol{x}$ is $\begin{bmatrix} 1 & x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$ to account for the constant term. Our constraints can be represented in a matrix A such that $A\boldsymbol{x} \leq \boldsymbol{b}$. After the transformation, $\boldsymbol{x}$ becomes $\boldsymbol{y} = y_0 \boldsymbol{x}$ and the objective becomes just the numerator, $\boldsymbol{c}_1^T \boldsymbol{y}$. We add the vector $-\boldsymbol{b}$ as the leftmost column of the matrix and set $\boldsymbol{b} = \vec{0}$. Finally, we add the denominator $\boldsymbol{c}_2^T$ as a row to the matrix, and 1 to $\boldsymbol{b}$ to turn the denominator into a constraint.

If we solve the new program, then we get solutions to $\boldsymbol{y}$. To get the final solution, we can compute $\boldsymbol{x}$ by just reversing the transformation, $\boldsymbol{x} = \frac{1}{y_0} \boldsymbol{y}$.

This process is called *Charnes-Cooper* and works great for continuous variables, but we quickly run into a problem for mixed-integer problems. Recall that we do the substitution $y_i = y_0 x_i$. y_0 is a continuous variable, and x_i is a discrete variable. What the heck is y_i then? If we let y_i be a continuous variable, that doesn't work because there's no guarantee that after we divide by y_0 we'll get an integer—y_i is free to take on many different values, not just multiples of y_0. Meanwhile if y_i is a discrete variable, that doesn't work because y_0 can take on many different continuous values. So y_i is ill-defined. In our particular case, however, x_i is a binary variable meaning it's only ever 0 or 1. If there's only two possibilities, can't we somehow force y_i to equal $y_0 x_i$?

6.2.2 Glover's Linearization

The general technique is called *Glover's linearization*, "linearization" because it converts a nonlinear term into a linear term. Suppose $\boldsymbol{x}$ is a vector of binary variables and $\boldsymbol{y}$ is a vector of continuous variables. We wish to create a new continuous variable $z_i = g_i(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{x}_i$, where $g_i(\boldsymbol{x}, \boldsymbol{y})$ is a linear function of the variables. We assume that it does not contain x_i since $x_i^2 = x_i$ ($0^2 = 0$, $1^2 = 1$) and that it does not contain constants (since multiplying by x_i would be linear). We define L to be the minimum value of g_i on the *relaxation* of the domain X, X^R, where the relaxation is the domain with the integer requirements removed, or "relaxed". We define U to be the maximum value of g_i on X^R. In order to enforce that z is the product of g_i and x_i, we enforce two constraints:

$$L_i x_i \leq z_i \leq U_i x_i \tag{1}$$

$$g_i(\boldsymbol{x}, \boldsymbol{y}) - U_i(1 - x_i) \leq z_i \leq g_i(\boldsymbol{x}, \boldsymbol{y}) - L_i(1 - x_i) \tag{2}$$

To see why this works, we can simply examine both cases. If $x_i = 0$ then the first constraint yields $0 \leq z_i \leq 0$, forcing $z_i = 0$. The second constraint is then $g_i(\boldsymbol{x}, \boldsymbol{y}) - U_i \leq z_i \leq g_i(\boldsymbol{x}, \boldsymbol{y}) - L_i$. $g_i(\boldsymbol{x}, \boldsymbol{y}) - U_i$ must be less than or equal to zero since U is the maximum value, and similarly, $g_i(\boldsymbol{x}, \boldsymbol{y}) - L_i$ must be greater than or equal to zero since L is the minimum value. $z_i = 0$ thus necessarily satisfies the second constraint.

If $x_i = 1$, then the second constraint yields $g_i(\boldsymbol{x}, \boldsymbol{y}) \leq z_i \leq g_i(\boldsymbol{x}, \boldsymbol{y})$, forcing $z_i = g_i(\boldsymbol{x}, \boldsymbol{y})$. The first constraint reduces to $L_i \leq z_i \leq U_i$, which is satisfied by the definition of L and U. Thus, the two constraints force $z_i = g_i(\boldsymbol{x}, \boldsymbol{y})x_i$ for any value of x_i.

6.2.3 Solving MILFPs

We can now solve mixed-integer linear fractional programs. We first apply Charnes-Cooper to obtain a non-linear integer program. We then apply Glover's linearization to form a mixed-integer linear program. While this technique is useful in the sense that we only need to solve one MILP, it significantly increases the problem size. If the original problem had I continuous variables, J binary variables, and K constraints, we add $\underbrace{J}_{\text{Glover's}} + \underbrace{1}_{\text{denominator}}$ new continuous variables and $\underbrace{3J}_{\text{Glover's}} + \underbrace{1}_{\text{denominator}}$ new constraints. We add 3 constraints instead of 4 because the denominator is necessarily positive, so $L = 0$ and thus $L_i x_i \leq z_i$ becomes $0 \leq z_i$, which is usually implicit in linear programming.

6.3 `$antidisablelist` Optimization and Expected Value

Before we can apply MILFPs to our problem of maximizing kakera, we must formulate the program in the first place. Before we formulate the problem, we introduce a new mechanic: *antidisable* lists. We can antidisable up to 500 series, where antidisabling a series means that it is no longer disabled. This is irrelevant if we want to maximize the number of characters disabled as we did in the previous section, but is now relevant for maximizing kakera value—it is conceivable that we disable a bundle with many low-value characters and then selectively antidisable the series that contain high-value characters.

We can implement this mechanic by introducing a new binary variable z_i which represents whether the ith series is disabled or not. We have that the sum of z_i is less than or equal to 500, and that $z_i \leq y_i$ since we can't antidisable the series if the series isn't disabled. Finally, we replace y_i in the objective with $y_i - z_i$ to "zero-out" antidisabled series. What is our objective, anyways? Instead of maximizing the number of characters disabled, we want to maximize kakera. However, our model for kakera value is incredibly nonlinear. The best we can do is use a heuristic, and the simplest is probably the expected value of the kakera distribution. This corresponds to maximizing $\mathrm{E}[X]$, which is of course not maximizing $\mathrm{E}[X_{30}]$, but a higher $\mathrm{E}[X]$ should mean higher character values overall. How do we formulate this objective of maximizing expected value? Let t_i be the total kakera value of the ith series. Then the expected value $\mathrm{E}[X] = \frac{\sum t_i(1-y_i)}{\sum w_i(1-y_i)}$ because y_i is whether a series is *disabled*, so $1 - y_i$ is 1 if it is active, and 0 if the series is disabled. With the objective complete, we make one last modification before formulating the MILFP. Recall that the original linear program never explicitly enforced that disabling a bundle disables the series it contains. That was fine for the old objective of maximizing characters, but in this case it might be advantageous to not disable—since it is essentially a free antidisable. We must therefore enforce that $y_i \geq x_j$ for each x_j that contains y_i.

name	variable
x_i	ith bundle disabled or not
y_i	ith series disabled or not
z_i	ith series antidisabled or not
s_i	size of the ith bundle
w_i	number of wa characters in the ith series
t_i	total kakera value of the ith series

Table 2: Variable names.

$$\text{maximize } \frac{\sum t_i(1-(y_i-z_i))}{\sum w_i(1-(y_i-z_i))} \quad \text{(expected value)} \qquad \text{(expected value program)}$$

subject to:

$$\sum x_i \leq K \quad \text{(can only pick } K = 10 \text{ bundles)}$$

$$\sum s_i x_i \leq 20000 \quad \text{(overlap limit)}$$

$$\sum z_i \leq 500 \quad \text{(antidisable limit)}$$

for each y_i:

$$\sum_{j|y_i \in x_j} x_j \geq y_i \quad \text{(if } y_i \text{ is disabled, at least one bundle disables it)}$$

$$y_i \geq x_j \quad \text{(disable } y_i \text{ if at least one bundle disables it)}$$

$$z_i \leq y_i \quad \text{(can't antidisable if series isn't disabled)}$$

$$x_i, y_j, z_k \in \{0,1\} \quad (x, y, \text{and } z \text{ are binary variables})$$

Before we blindly apply our MILFP solving technique we notice that x_i does not appear in the objective. In the continuous case, we essentially discard the old variables and replace all of them with the transformed variables, but in the mixed case we keep the old binary variables to enforce the constraints on the transformed variables. We therefore save constraints and variables if we simply don't multiply inequalities on both sides by the denominator, and because we skip the multiplication we don't need to transform x_i at all. A bit of ambiguity is whether we apply constraints like $\sum z_i \leq 500$ to the transformed z_i or the binary z_i. z_i (and y_i) are essentially unconstrained if the constraints are written entirely in the form of their transformed versions, and take their values through the constraints in Glover's linearization. It is therefore more consistent with x_i, simpler, and more efficient to enforce the constraints on the binary versions rather than the transformed versions.

We denote the continuous denominator variable for Charnes-Cooper as d and we denote the transformed variables $y'_i = dy_i$ and $z'_i = dz_i$. We must also determine the upper and lower bounds of d for Glover's linearization. The denominator is one over the sum of the characters of the series that are not disabled, so it is nonnegative (there would have to be infinite characters to make $d = 0$) and its largest value is when the number of characters is the smallest, when as many characters as possible are disabled. There is a hard limit of 90% of characters disabled, so d is bounded, and a simple upper bound is 1. The transformed program will look very similar to the base program:

$$\text{maximize } \sum t_i(d - (y'_i - z'_i)) \quad \text{(numerator of expected value)} \quad \text{(linearized)}$$

subject to:

$$\sum w_i(d - (y'_i - z'_i)) = 1 \quad \text{(denominator of expected value)}$$

$$\sum x_i \leq K \quad \text{(can only pick } K = 10 \text{ bundles)}$$

$$\sum s_i x_i \leq 20000 \quad \text{(overlap limit)}$$

$$\sum z_i \leq 500 \quad \text{(antidisable limit)}$$

for each y_i:

$$\sum_{j|y_i \in x_j} x_j \geq y_i \quad \text{(if } y_i \text{ is disabled, at least 1 bundle disables it)}$$

$$y_i \geq x_j \quad \text{(disable } y_i \text{ if at least 1 bundle disables it)}$$

$$z_i \leq y_i \quad \text{(can't antidisable if series isn't disabled)}$$

for each y_i and z_i, call the variable v_i:

$$Lv_i \leq v'_i \leq Uv_i \quad \text{(1st condition of Glover's)}$$

$$d - U(1 - v_i) \leq v'_i \leq d - L(1 - v_i) \quad \text{(2nd condition of Glover's)}$$

$$x_i, y_j, z_k \in \{0, 1\} \quad (x, y, \text{ and } z \text{ are binary variables})$$

$$d, y'_i, z'_j \in [L, U] \quad \text{(continuous variables)}$$

The program is pretty sensitive to L and U. It is simplest to pick $L = 0$ since that removes a constraint (linear programs implicitly have nonnegative constraints). How does one determine U? We can simply use the definition, the largest value on the relaxation. If we relax the problem we get a linear program, whose objective is to *minimize* $\sum w_i(1-(y_i-z_i))$ since d is one over that expression. Since linear programming is polynomial time, it will run much quicker than the MILFP. Even with the tight upper bound of $U = 0.0002$, the MILP does not converge in any reasonable amount of time. One must therefore restrict the running time and forgo convergence. It is also helpful to reduce the number of variables, for example, to restrict the bundles being disabled to the top 100 largest bundles as the smaller bundles necessarily have less of an effect.

Another trick is to re-write the constraint that $y_i \geq x_j$. It looks innocent because of the notational trick: I mean that y_i needs to be greater than *each* x_j such that $y_i \in x_j$. In the worst case each series could add up to the number of bundles to the number of constraints. The intuition is that we want to force y_i to be 1 if any x_j is 1, or if $x_1 + x_2 + \cdots + x_n$ is greater than 0 (addition is essentially bitwise OR). We can't just enforce that $y_i \geq x_1+x_2+\cdots+x_n$ because if multiple x_j are on, y_i can only be at most 1. So we can divide by the same thing to "normalize" the addition into a boolean. We then enforce $y_i \geq \frac{x_1+x_2+\cdots+x_n}{x_1+x_2+\cdots+x_n}$ which is not equivalent to enforcing that $y_i \geq 1$ because of the case where all x_j are 0. To see the difference, multiply across to linearize the constraint: $(x_1 + x_2 + \cdots + x_n)y_i = x_1 + x_2 + \cdots + x_n$. If each x_j is 0, then the constraint reduces to $0 = 0$, which is trivially satisfied. The other constraint that $\sum x_j \geq y_i$ then forces y_i to be 0, which must be true if none of the bundles which contain it are disabled. If at least one of the x_j's is 1, then y_i is forced to be 1 to make $y_i \geq \frac{x_1+x_2+\cdots+x_n}{x_1+x_2+\cdots+x_n}$ true. Meanwhile the second constraint is now redundant as it reduces to $\sum x_j \geq 1$, which is satisfied as we know at least one of the x_j is 1. Although the new constraint works to enforce the behavior we want, the product between y_i and the sum of x_j's then needs to be linearized with Glover's, which we can do all at once instead of for each x_j:

$$g_i(\boldsymbol{x}, \boldsymbol{y}) = x_1 + x_2 + \cdots + x_n \tag{3}$$

$$L_i y_i \leq y_i' \leq U_i y_i \tag{4}$$

$$g_i(\boldsymbol{x}, \boldsymbol{y}) - U_i(1 - y_i) \leq y_i' \leq g_i(\boldsymbol{x}, \boldsymbol{y}) - L_i(1 - y_i) \tag{5}$$

$$y_i' \geq g_i(\boldsymbol{x}, \boldsymbol{y}) \tag{6}$$

L is clearly 0 if all x_j are 0 and a simple upper bound for U is simply the number of x_j which contain y_i, n. Using these new bounds, we can simplify the 5 inequalities considerably. The left half of (4), $L_i y_i \leq y_i'$, becomes $0 \leq y_i'$, which is again implicit in the definition of y_i', so it can be removed. For the left half of (5), we consider the two cases of y_i. If $y_i = 0$ then $y_i' = 0$ from the right half of (4) and $g_i(\boldsymbol{x}, \boldsymbol{y}) - U_i \leq 0$ follows from the definition of U. If $y_i = 1$, then the inequality reduces to $g_i(\boldsymbol{x}, \boldsymbol{y}) \leq y_i'$ which is exactly (6). Thus, because it is redundant in either case, we can remove this constraint as well. Finally, we notice that if $L = 0$, then the right side of (5) reduces to $g_i(\boldsymbol{x}, \boldsymbol{y}) \leq y_i'$, which can be combined with (6) to yield $y_i' = g_i(\boldsymbol{x}, \boldsymbol{y})$. Since this is an equality constraint with just one variable, we can avoid creating a new continuous variable for y_i' by just substituting its value wherever y_i' occurs, in this case just $y_i' \leq U_i y_i$.

Recall that we started by adding possibly n new constraints per series, where n is the number of bundles which contain the series. Using the re-write and Glover's linearization, we turned $O(n)$ into $O(1)$ by adding at most 5 constraints and 1 continuous variable. Finally, with the observations pruning extraneous constraints, we add just 1 constraint:

$$g_i(\boldsymbol{x}, \boldsymbol{y}) \leq U_i y_i$$

After running for 2 hours on Snowy, the linear program found an expected value of 75.679 compared to the original 61.702 and a model expected value of 429.839 compared to 322.371. A 14 kakera improvement in the kakera distribution is nothing extraordinary, but that difference is heavily magnified by the model, increasing the average value per claim by more than 100 kakera.

6.3.1 Linear Regression

While expected value is a decent heuristic, can we do better? Our objective must be linear, because we are using linear programming. Is there a way to turn a nonlinear function into a linear function? Well, the best we can do is to fit a line. We fit this line with *linear regression*—my derivation can be found here.

Our objective is a mapping from the active series to the expected value of the model. We can represent the active series as a bitstring of whether a series is active or not, so the dimensionality of the input is the number of series. We randomly and uniformly[8] generate a bit string by generating each bit at a time, where each bit has a 0.5 chance to be 1 and a 0.5 chance to be off. We then extract a random variable from the resulting character data and feed the random variable to our model, which computes the expected value. Doing this multiple times to generate a dataset, we fit linear regression to the data which gives us a linear function representing the relationship between the series variable and the expected value. We can then use this line as the objective of our linear program.

The benefits over the MILFP for expected value is that our linear regression might be more aligned with the model expected value and that we do not increase the problem size. Despite the significantly reduced problem size, it still does not converge in time. In terms of the results, I generated a dataset with 10,000 rows. The linear regression has a R^2 of 0.953 on the training dataset but only a 0.586 R^2 on the testing dataset of size 1,000, suggesting a lack of generalization.[9] As a result of this lack of alignment, the expected value of the kakera distribution is actually lower and the model expected value is approximately the same.

[8] What does it mean to generate something "uniformly"? Each thing in the space must have an equal probability of being picked. There are 2^n bitstrings of length n, and if we generate bit by bit, with each bit having a 0.5 chance to be 1, then each string has a $\frac{1}{2} \cdot \frac{1}{2} \cdot \ldots \cdot \frac{1}{2} = \frac{1}{2^n}$ chance of being generated.

[9] I am aware that a higher score on the training dataset relative to the testing dataset is a classic sign of *overfitting*. However, in this situation the linear regression is by definition underfitting—the relationship is highly nonlinear, so a linear model is not the right choice.

6.4 Conclusion

I began this section optimistically, stating that NP-hard is not a death sentence—problems can still be tractable in the average case. My optimism is tempered by the lack of convergence I have witnessed for the latter programs. To summarize, we will compare the sizes of each of the programs developed. If B is the number of bundles and S is the number of series, then:

name	binary variables	continuous variables	constraints
cardinality	$(B+S)+S$ $B+2S$	0 0	$1+1+S$ $S+2$
fractional program	$[B+2S]+S$ $B+3S$	0 0	$[S+2]+1+S+S$ $3S+3$
linearized program	$B+3S$ $B+3S$	$2S+1$ $2S+1$	$[3S+3]+3(2S)+1$ $9S+4$

Table 3: Program sizes.

The sizes grow quite quickly from the cardinality program to the final MILFP. Likewise, the cardinality program converges in less than 10 seconds while I have no idea how long the expected value MILFP takes to converge—all I can say is that it's longer than 24 hours. Would the parametric algorithm instead of the reformulation-linearization explored in this section be more efficient? I encourage the reader to see for themselves. Whatever the case, the problem is rich for future exploration—the optimization of an incomplete *heuristic* hasn't even been solved! Can the connection between optimizing disablelists while taking into account the probabilistic model ever be completed in full? I wouldn't have written 50 pages if I thought that the task was truly impossible. □

7 Extensions and Future Work

Despite the length of this paper, there remains many avenues for future research.

1. *A proper way to maximize expected value.* In order to optimize disablelists, we use expected value of the kakera distribution as a heuristic for the expected value of the model. Can we get better performance with quadratic programming or convex optimization techniques? Deep learning?

2. *A provably optimal model over a known timescale.* As stated before, if the timescale is known then the optimal model changes (because expected value is optimal over an infinite timescale). Can we create a model that is optimal for short timescales?

3. *Empirically inferring taste distributions.* People do not act to maximize expected kakera value. They have characters they like, which they might claim without regard to its quantitative value. Suppose we assume that these people are not simply irrational but have a different value distribution, a phenomenon often called *taste*. Can we infer a person's taste distribution from their actual claiming behavior?

4. *Implementing more Mudae mechanics.* Wishlists, badges, kakera reacts, etc.

5. *A Discord bot.* While self-botting is against the terms of service, a discord bot which one could invoke for advice would be nicer than interacting with the CLI.

6. *A class for random variables.* It would be cleaner to create a `RandomVariable` class which contained all the helper methods like PMF, CMF, expectation, etc. One could then instantiate this class by providing a list of values and their probabilities. That would be nicer than carrying around lists everywhere and managing functions.

8 Sources

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2. My convolution and the FFT lecture
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4. Maximum set cover
5. Python mixed-integer linear programming library
6. Duality in linear programming
7. Ratios in linear programming
8. Solving mixed-integer linear fractional programs
9. Glover's linearization
10. Relaxation

9 Appendix

9.1 Secretary Problem

Mudae is very related to the secretary problem. Suppose you are a secretary interviewing job applicants. You have n candidates and there is a clear ranking among the candidates, e.g. after interviewing a candidate you can tell how they rank amongst the candidates you've already interviewed. The twist is that you must choose to accept or reject a candidate immediately after their interview. What should be your strategy to maximize the probability of picking the best candidate?

It can be shown that the optimal strategy is of the form "always reject the first r candidates, then pick the first candidate who is better than all the candidates you've rejected." Interestingly enough, r approaches $\frac{n}{e}$ as n approaches ∞, so you should reject the first $\frac{1}{e}$ fraction of candidates or about the first 36.8%. The probability of picking the best candidate is also $\frac{1}{e}$, regardless of the size of n.

How is this problem related to Mudae? Both are based off the immediate decision making for a given candidate. Consider the following two solutions for the secretary problem: one picks the best candidate with a 50% chance, but picks the absolute worst candidate the rest of the time. The other picks the best candidate with a 49% chance, and picks the second best the rest of the time. Is the first strategy really better than the second strategy? If one weights the candidates by some quantitative value with a known distribution and wants to maximize the *expected value* rather than simply the maximum value, then the problem reduces to Mudae.

9.2 Conditional Probability

Definition 9.1. The probability that an event A happens *conditional* on B is denoted $p(A|B)$, i.e. the probability A happens "given" B happens. This should not be confused for $p(A \cap B)$, the probability A happens *and* B happens.

Corollary 9.0.1.

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

Proof. We will use an informal (and therefore loose) definition of probability as the number of ways for something to occur over the total number of ways. Since we know B occurs, the total number of ways is $p(B)$. The number of ways A occurs is the number of events where A occurs *and* B occurs, because we know B occurs. □

One way to picture this is to imagine a Venn diagram in a rectangle, where the rectangle is the universe of all possible events, one circle is A, and the other B. If we restrict our universe to B since we know B occurs, then the ways A occurs is the intersection of A and B, $A \cap B$. The ways B occurs is just B's circle, $p(B)$.

Theorem 9.1. *Bayes' Theorem, i.e. switching conditional probabilities:*

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

Proof. From the definition of conditional probability we know

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

Multiplying by $p(B)$ on both sides,

$$p(A|B)p(B) = p(A \cap B)$$

Since we know $A \cap B = B \cap A$,

$$p(B \cap A) = p(A|B)p(B) = p(B|A)p(A)$$

where the latter identity comes from the original identity. Thus,

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$ □

One way to interpret this theorem is suppose A is our hypothesis and B is a new piece of data. Then $p(A|B)$ is how confident we are in the hypothesis given the new piece of data, $p(B|A)$ is how probable the data is under our hypothesis, $p(A)$ is the *prior* probability in our hypothesis (how sure we are in the hypothesis before piece of data), and $p(B)$ is how probable the data is overall (which, in practice, is often intractable: how do we compute the probability over *all* hypotheses?).

9.3 "General" Rolls Model

We have two key assumptions:

1. Use at most one `$rolls` per claim cycle.
2. k^* is less than $\mathrm{E}[X_B]$

The second assumption can be checked empirically and is only for making introspection easier. Can we formulate an optimal model with an unlimited usage of `$rolls` per claim?

The observation is that we want to avoid low kakera values, regardless of when they occur. For example, the minimum possible value is 32. Suppose we are about to claim a 32 value character, and use `$rolls` to avoid this fate. However, we get unlucky and all 10 of our new rolls were 32 yet again. Should we use `$rolls` again, or at some other time? Well, in what other situation would it be as beneficial to avoid the minimum possible value? Ignoring the fact that we've already used `$rolls`, clearly we should use it again.

The probability of being about to claim $k = 32$ is the probability we're at the last layer times the probability of rolling k, $p(\text{emit at } 1)p_Z(k)$. Then the chance we use `$rolls` once is $p(1)p_Z(k)(1 - p_Z(k))$ since we if we use it exactly once, we first roll k then roll anything higher than k, which is anything except k. The chance we use it twice is $p(1)p_Z(k)^2(1 - p_Z(k))$, three times $p(1)p_Z(k)^3(1 - p_Z(k))$, and so on. This is a geometric random variable with probability of success $1 - p_Z(k)$ (in this case "success" is *not* rolling k since that's when we break out of using `$rolls`). The expected value is then $\frac{p(\text{emit at1})p_Z(k)}{1-p_Z(k)}$, which is how many times we expect to use `$rolls` if we use it only on the minimum value. If this is less than $\frac{1}{8}$, then we can also use `$rolls` on the second smallest value. In general, we find the largest k^* such that $\frac{p(\text{emit at1})F_Z(k^*)}{1-F_Z(k^*)}$ is smaller than $\frac{1}{8}$ (since the chance we "enter" a cycle of using `$rolls` is $p(1)F_Z(k^*)$ and the chance we break out is if we roll something higher than k^*, a $1 - F_Z(k^*)$ chance). We find this k^* with binary search like the standard rolls model, and if we binary search on indexes the time is $O(\log n)$ rather than binary searching on values, $O(\log^2 n)$ because of the $O(\log n)$ cost to compute the CMF at a value not in the range.

This has the effect of lowering k^* relative to the standard rolls model, 68 versus 72. This is because more probability mass is concentrated in the lower kakera values, since we can use `$rolls` a theoretically infinite times on each kakera value smaller than k^*. It also slightly raises the expected value since we use `$rolls` on the worst situations, 322.371 for the general model versus 322.036 for the standard model.

Unfortunately, the actual game limits `$rolls` to once per interval, so this model is simply an interesting demonstration of geometric random variables.